The generalized Abel-Plana formula with applications to Bessel functions and Casimir effect

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Abstract

One of the most efficient methods for the evaluation of the vacuum expectation values for physical observables in the Casimir effect is based on using the Abel-Plana summation formula. This enables to derive the renormalized quantities in a manifestly cutoff independent way and to present them in the form of strongly convergent integrals. However, applications of the Abel-Plana formula, in its usual form, are restricted by simple geometries when the eigenmodes have a simple dependence on quantum numbers. The author generalized the Abel-Plana formula which essentially enlarges its application range. Based on this generalization, formulae have been obtained for various types of series over the zeros of combinations of Bessel functions and for integrals involving these functions. It has been shown that these results generalize the special cases existing in literature. Further, the derived summation formulae have been used to summarize series arising in the direct mode summation approach to the Casimir effect for spherically and cylindrically symmetric boundaries, for boundaries moving with uniform proper acceleration, and in various braneworld scenarios. This allows to extract from the vacuum expectation values of local physical observables the parts corresponding to the geometry without boundaries and to present the boundary-induced parts in terms of integrals strongly convergent for the points away from the boundaries. As a result, the renormalization procedure for these observables is reduced to the corresponding procedure for bulks without boundaries. The present paper reviews these results. We also aim to collect the results on vacuum expectation values for local physical observables such as the field square and the energy-momentum tensor in manifolds with boundaries for various bulk and boundary geometries.

Keywords: Abel-Plana summation formula; Bessel functions; Casimir effect

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Contents

1 Introduction

In many physical problems we need to consider the model on background of manifolds with boundaries on which the dynamical variables satisfy some prescribed boundary conditions. In quantum field theory the imposition of boundary conditions leads to the modification of the spectrum for the zero-point fluctuations and results in the shift in the vacuum expectation values for physical observables. In particular, vacuum forces arise acting on the constraining boundary. This is the familiar Casimir effect [1]. The Casimir effect is among the most interesting consequences of quantum field theory and is essentially the only macroscopic manifestation of the nontrivial properties of the physical vacuum. Since the original work by Casimir many theoretical and experimental works have been done on this problem, including various bulk and boundary geometries and boundary conditions (see, for instance, [2, 3, 4, 5, 6, 7] and references therein). Many different approaches have been used: direct mode summation method, Green function formalism, multiple scattering expansions, heat-kernel series, zeta function regularization technique, etc. Advanced field-theoretical methods have been developed for Casimir calculations during the past years [8, 9, 10]. From a general theoretical point of view the main point here is the unique separation and subsequent removing of the divergences. Within the framework of the mode summation method in calculations of the expectation values for physical observables, such as energy-momentum tensor, one often needs to sum over the values of a certain function at integer points, and then subtract the corresponding quantity for unbounded space (usually presented in terms of integrals). Practically, the sum and integral, taken separately, diverge and some physically motivated procedure to handle the finite result, is needed (for a discussion of different methods to evaluate this finite difference see [11]). For a number of geometries one of the most convenient methods to obtain such renormalized values of the mode sums is based on the use of the Abel-Plana summation formula (APF) [12, 13, 14, 15, 16, 17]. The development of this formula dates back to Plana in 1820 [18] and further has been reconsidered by Abel [19], Cauchy [20] and Kronecker [21]. The history of the APF is discussed in detail in [22] where interesting applications are given as well. Some formulae for the gamma and zeta functions are obtained as direct consequences of the APF (see, for instance, [15, 23]). This summation formula significantly and consistently improves the convergence and accuracy for the slowly convergent series (see, for instance, [24]). The summation method based on the APF is so efficient that was considered for the development of software for special functions [25]. In [26] the APF has been used for the renormalization of the scalar field energy-momentum tensor on backgrounds of various Friedmann cosmological models. Further applications of the APF in physical problems related to the Casimir effect for flat boundary geometries and topologically non-trivial spaces with corresponding references can be found in [2, 3]. The application of the APF in these problems allows (i) to extract in a cutoff independent way the Minkowski vacuum part and (ii) to obtain for the renormalized part strongly convergent integrals useful, in particular, for numerical calculations.

The applications of the APF in its usual form is restricted to the problems where the eigenmodes have simple dependence on quantum numbers and the normal modes are explicitly known. For the case of curved boundaries and for mixed boundary conditions the normal modes are given implicitly as zeroes of the corresponding eigenfunctions or their combinations. The necessity for a generalization of the Abel-Plana summation procedure arises already in the case of plane boundaries with Robin or non-local boundary conditions, where the eigenmodes are given implicitly as solutions of transcendental equation. In problems with spherically and cylindrically symmetric boundaries the eigenmodes are the zeroes of the cylinder functions and the expectation values of physical observables contain series over these zeroes. To include more general class of problems, in [27] the APF has been generalized (see also [28]). The generalized version

contains two meromorphic functions and the APF is obtained by specifying one of them (for other generalizations of the APF see [3, 11, 29]). Applying the generalized formula to Bessel functions, in [27, 28] summation formulae are obtained for the series over the zeros of various combinations of these functions (for a review with physical applications see also Ref. [30, 31]). In particular, formulae for the summation of the Fourier-Bessel and Dini series are derived. It has been shown that from the generalized formula interesting results can be obtained for infinite integrals involving Bessel functions. The summation formulae derived from the generalized Abel-Plana formula have been applied for the evaluation of the vacuum expectation values for local physical observables in the Casimir effect for plane boundaries with Robin or non-local boundary conditions [32, 33], for spherically [34, 35, 36, 37, 38] and cylindrically symmetric [39, 40, 41, 42] boundaries on the Minkowski bulk for both scalar and electromagnetic fields. As in the case of the APF, the use of the generalized formula allows to extract in a manifestly cutoff independent way the contribution of the unbounded space and to present the boundary-induced parts in terms of exponentially converging integrals. The case of cylindrical boundaries in topologically nontrivial background of the cosmic string is considered in [43, 44]. By making use of the generalized Abel-Plana formula, the vacuum expectation values of the field square and the energy-momentum tensor in closely related but more complicated geometry of a wedge with cylindrical boundary are investigated in [45, 46, 47] for both scalar and electromagnetic fields. In [48, 49, 50] summation formulae for the series over the zeroes of the modified Bessel functions with an imaginary order are derived by using the generalized Abel-Plana formula. This type of series arise in the evaluation of the vacuum expectation values induced by plane boundaries uniformly accelerated through the Fulling-Rindler vacuum. In all considered examples the background spacetime is flat.

For curved backgrounds with boundaries the application of the generalized Abel-Plana formula extracts from the expectation values the parts which correspond to the polarization of the vacuum in the situation without boundaries and the boundary induced parts are presented in terms of integrals which are convergent for points away from the boundaries. As a result, the renormalization is necessary for the boundary-free parts only and this procedure is the same as that in quantum field theory without boundaries. Examples with both bulk and boundary contributions to the vacuum polarizations are considered in [51, 52, 53, 54, 55]. In these papers the background spacetime is generated by a global monopole and the one-loop quantum effects for both scalar and fermionic fields are investigated induced by spherical boundaries concentric with the global monopole. Another class of problems where the application of the generalized Abel-Plana formula provides an efficient way for the evaluation of the vacuum expectation values is considered in [56, 57]. In these papers braneworld models with two parallel branes on anti-de Sitter bulk are discussed. The corresponding mode-sums for physical observables bilinear in the field contain series over the zeroes of cylinder functions which are summarized by using the generalized Abel-Plana formula. The geometry of spherical branes in Rindler-like spacetimes is considered in [58]. In [59] from the generalized Abel-Plana formula a summation formula is derived over the eigenmodes of a dielectric cylinder and this formula is applied for the evaluation of the radiation intensity from a point charge orbiting along a helical trajectory inside the cylinder.

The present paper reviews these results and is organized as follows. In Section 2 the generalized Abel-Plana formula is derived and it is shown that the APF is obtained by a special choice of one of the meromorphic functions entering in the generalized formula. It is indicated how to generalize the APF for functions having poles. The applications of the generalized formula to the Bessel functions are considered in Section 3. We derive two formulae for the sums over zeros of the function $AJ_\nu(z) + BzJ'_\nu(z)$, where $J_\nu(z)$ is the Bessel function. Specific examples of applications of the general formulae are considered. In Section 4, by special choice

of the function $g(z)$ summation formulae are derived for the series over zeros of the function $J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$ and similar combinations with Bessel functions derivatives. Special examples are considered. In Section 5 we consider the application of the generalized Abel-Plana formula to the series over the zeros of the modified Bessel functions with an imaginary order. The applications to the integrals involving Bessel functions and their combinations are discussed in Section 6. A number of interesting results for these integrals are presented. Specific examples of applying these general formulae are described in Section 7. In Section 8, by using the generalized Abel-Plana formula two theorems are proved for the integrals involving the function $J_\nu(z)Y_\mu(\lambda z) - J_\mu(\lambda z)Y_\nu(z)$ and their applications are considered.

In the second part of the paper we consider the applications of the generalized Abel-Plana formula to physical problems, mainly for the investigation of vacuum expectation values of physical observables on manifolds with boundaries of different geometries and boundary conditions. First, in Section 9 we describe the general procedure for the evaluation of the vacuum expectation values of physical observables bilinear in the field operator. Applications of the APF for the evaluation of the the vacuum expectation values of the field square and the energy-momentum tensor for a scalar field in the spacetime with topology $R^D \times S^1$ and for the geometry of two parallel Dirichlet and Neumann plates are given in Section 10. In Section 12, the summation formulae over the zeroes of combinations of the cylinder functions are applied for the investigation of the scalar vacuum densities induced by spherical boundaries on background of the global monopole spacetime. The corresponding problem for the fermionic field with bag boundary conditions is considered in Section 13. In the latter case the eigenmodes are the zeroes of the more complicated combination of the Bessel function and its derivative. Summation formulae for series over these zeroes are derived from the generalized Abel-Plana formula. The application to the electromagnetic Casimir effect for spherical boundaries in background of the Minkowski spacetime is given in Section 14. Further we consider problems with cylindrical boundaries. In Section 15 we study the vacuum polarization effects for a scalar field induced by a cylindrical shell in the cosmic string spacetime assuming Robin boundary condition on the shell. The corresponding problem for the electromagnetic field and perfectly conducting cylindrical boundary is considered in Section 16. Section 17 is devoted to the investigation of the vacuum densities in the region between two coaxial cylindrical surfaces in the Minkowski spacetime for both scalar and electromagnetic fields. In Sections 18 and 19 we consider the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor for scalar and electromagnetic fields induced by plane boundaries uniformly accelerated through the Fulling-Rindler vacuum. The application of the generalized Abel-Plana formula for the evaluation of the vacuum densities in braneworld scenarios is described in Section 20 for parallel branes on background of anti-de Sitter spacetime and in Section 21 for spherical branes in Rindler-like spacetimes. In Section 22, a summation formula for the series over the eigenmodes of a dielectric cylinder is derived and this formula is applied for the investigation of the radiation intensity from a charged particle moving along a helical orbit inside the cylinder. Section 23 concludes the main results of the paper.

2 Generalized Abel-Plana formula

Let $f(z)$ and $g(z)$ be meromorphic functions for $a \leqslant x \leqslant b$ in the complex plane $z = x + iy$. Let us denote by $z_{f,k}$ and $z_{g,k}$ the poles of $f(z)$ and $g(z)$ in the strip $a < x < b$. We assume that $\mathrm{Im}\, z_{f,k} \neq 0$ (see, however, the Remark to Lemma).

Lemma. *If functions $f(z)$ and $g(z)$ satisfy condition*

$$\lim_{h\to\infty}\int_{a\pm ih}^{b\pm ih}[g(z)\pm f(z)]\,dz=0, \tag{2.1}$$

then the following formula takes place

$$\int_a^b f(x)dx=R[f(z),g(z)]-\frac{1}{2}\int_{-i\infty}^{+i\infty}[g(u)+\sigma(z)f(u)]_{u=a+z}^{u=b+z}\,dz,\ \sigma(z)\equiv\mathrm{sgn}(\mathrm{Im}\,z), \tag{2.2}$$

where

$$R[f(z),g(z)]=\pi i\left[\sum_k \underset{z=z_{g,k}}{\mathrm{Res}}\,g(z)+\sum_k\sigma(z_{f,k})\underset{z=z_{f,k}}{\mathrm{Res}}\,f(z)\right]. \tag{2.3}$$

Proof. Consider a rectangle C_h with vertices $a\pm ih$, $b\pm ih$, described in the positive sense. In accordance to the residue theorem

$$\int_{C_h}dz\,g(z)=2\pi i\sum_k\underset{z=z_{g,k}}{\mathrm{Res}}\,g(z), \tag{2.4}$$

where the rhs contains the sum over poles within C_h. Let C_h^+ and C_h^- denote the upper and lower halves of this contour. Then one has

$$\int_{C_h}dz\,g(z)=\sum_{\alpha=+,-}\left\{\int_{C_h^\alpha}dz\,[g(z)+\alpha f(z)]-\alpha\int_{C_h^\alpha}dz\,f(z)\right\}. \tag{2.5}$$

By the same residue theorem

$$\int_{C_h^-}dz\,f(z)-\int_{C_h^+}dz\,f(z)=2\int_a^b f(x)dx-2\pi i\sum_k\sigma(z_{f,k})\underset{z=z_{f,k}}{\mathrm{Res}}\,f(z). \tag{2.6}$$

Then

$$\int_{C_h^\pm}dz\,[g(z)\pm f(z)]=\pm\int_0^{\pm ih}dz\,[g(u)\pm f(u)]_{u=a+z}^{u=b+z}\mp\int_{a\pm ih}^{b\pm ih}dz\,[g(z)\pm f(z)]. \tag{2.7}$$

Combining these results and allowing $h\to\infty$ in (2.4) one obtains formula (2.2). ■

If the functions $f(z)$ and $g(z)$ have poles with $\mathrm{Re}\,z_{j,k}=a,b$ $(j=f,g)$ the contour has to pass round these points on the right or left, correspondingly.

Remark. Formula (2.2) is also valid when the function $f(z)$ has real poles $z_{f,n}^{(0)}$, $\mathrm{Im}\,z_{f,n}^{(0)}=0$, in the region $a<\mathrm{Re}\,z<b$ if the main part of its Laurent expansion near these poles does not contain even powers of $z-z_{f,n}^{(0)}$. In this case on the left of formula (2.2) the integral is meant in the sense of the principal value, which exists as a consequence of the above mentioned condition. For brevity let us consider the case of a single pole $z=z_0$. One has

$$\begin{aligned}\int_{C_h^-}dz\,f(z)-\int_{C_h^+}dz\,f(z)&=2\int_a^{z_0-\rho}dz\,f(z)+2\int_{z_0+\rho}^b dz\,f(z)\\ -2\pi i\sum_k\sigma(z_{f,k})\underset{z=z_{f,k}}{\mathrm{Res}}\,f(z)&+\int_{\Gamma_\rho^+}dz\,f(z)+\int_{\Gamma_\rho^-}dz\,f(z),\end{aligned} \tag{2.8}$$

with contours Γ_ρ^+ and Γ_ρ^- being the upper and lower circular arcs (with center at $z = z_0$) joining the points $z_0 - \rho$ and $z_0 + \rho$. By taking into account that for odd negative l

$$\int_{\Gamma_\rho^+} dz\, (z - z_0)^l + \int_{\Gamma_\rho^-} dz\, (z - z_0)^l = 0, \tag{2.9}$$

in the limit $\rho \to 0$ we obtain the required result. ■

In the following, on the left of (2.2) we will write p.v.$\int_a^b dx\, f(x)$, assuming that this integral converges in the sense of the principal value. As a direct consequence of Lemma one obtains [27]:

Theorem 1. *If in addition to the conditions of Lemma one has*

$$\lim_{b\to\infty} \int_b^{b\pm i\infty} dz\, [g(z) \pm f(z)] = 0, \tag{2.10}$$

then

$$\lim_{b\to\infty} \left\{ \text{p.v.} \int_a^b dx\, f(x) - R[f(z), g(z)] \right\} = \frac{1}{2} \int_{a-i\infty}^{a+i\infty} dz\, [g(z) + \sigma(z) f(z)], \tag{2.11}$$

where on the left $R[f(z), g(z)]$ *is defined as (2.3),* $a < \operatorname{Re} z_{f,k}, \operatorname{Re} z_{g,k} < b$, *and the summation goes over poles* $z_{f,k}$ *and* $z_{g,k}$ *arranged in order* $\operatorname{Re} z_{j,k} \leqslant \operatorname{Re} z_{j,k+1}$, $j = f, g$.

Proof. It is sufficient to insert in general formula (2.2) $b \to \infty$ and to use the condition (2.10). The order of summation in $R[f(z), g(z)]$ is determined by the choice of the integration contour C_h and by limiting transition $b \to \infty$. ■

We will refer formula (2.11) as generalized Abel-Plana formula (GAPF) as for $b = n + a$, $0 < a < 1$,

$$g(z) = -if(z) \cot \pi z, \tag{2.12}$$

with an analytic function $f(z)$, from (2.11) the Abel-Plana formula (APF) [12, 13, 14] is obtained in the form

$$\begin{aligned} \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \int_a^{n+a} dx\, f(x) \right] &= \frac{1}{2i} \int_a^{a-i\infty} dz\, f(z)(\cot \pi z - i) \\ &\quad - \frac{1}{2i} \int_a^{a+i\infty} dz\, f(z)(\cot \pi z + i). \end{aligned} \tag{2.13}$$

Another form of the APF, given in [15, 16], is obtained directly from formula (2.2) taking the function $g(z)$ in the form (2.12) with an analytic function $f(z)$ and $a = m$, $b = n$ with integers m and n. In this case the points $z = m$ and $z = n$ are poles of the function $g(z)$ and in the integral on the right of formula (2.2) they should be avoided by small semicircles from the right and from the left respectively. The integrals over these semicircles give the contribution $-f(m) - f(n)$ and one obtains the formula

$$\begin{aligned} \sum_{k=m}^{n} f(k) &= \int_m^n dx\, f(x) + \frac{1}{2} f(m) + \frac{1}{2} f(n) \\ &\quad + i \int_0^\infty dz\, \frac{f(m+iz) - f(n+iz) - f(m-iz) + f(n-iz)}{e^{2\pi z} - 1}. \end{aligned} \tag{2.14}$$

Expanding the functions in the numerator of the last integral over z and interchanging the order of the summation and integration, the integral is expressed in terms of the Bernoulli numbers B_{2j}. As a result, from (2.14) we obtain the formula

$$\sum_{k=m}^{n} f(k) = \int_{m}^{n} dx\, f(x) + \frac{1}{2}f(m) + \frac{1}{2}f(n) - \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\left[f^{(2j-1)}(m) - f^{(2j-1)}(n)\right], \quad (2.15)$$

which is a special case of the Euler-Maclaurin formula [16] (for a relationship between the APF and Euler-Maclaurin summation formula see also [60]).

A useful form of (2.13) may be obtained performing the limit $a \to 0$. By taking into account that the point $z = 0$ is a pole for the integrands and therefore has to be circled by arcs of the small circle C_ρ on the right and performing $\rho \to 0$, one obtains

$$\sum_{k=0}^{\infty} f(k) = \int_{0}^{\infty} dx\, f(x) + \frac{1}{2}f(0) + i\int_{0}^{\infty} dx\, \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}. \quad (2.16)$$

Note that, now condition (2.1) is satisfied if

$$\lim_{y\to\infty} e^{-2\pi|y|}|f(x+iy)| = 0, \quad (2.17)$$

uniformly in any finite interval of x. Formula (2.16) is the most frequently used form of the APF in physical applications. Another useful form (in particular for fermionic field calculations) to sum over the values of an analytic function at half of an odd integer points is obtained from (2.11) taking

$$g(z) = if(z)\tan \pi z, \quad (2.18)$$

with an analytic function $f(z)$. This leads to the summation formula (see also [2, 3])

$$\sum_{k=0}^{\infty} f(k+1/2) = \int_{0}^{\infty} dx\, f(x) - i\int_{0}^{\infty} dx\, \frac{f(ix) - f(-ix)}{e^{2\pi x} + 1}. \quad (2.19)$$

By adding to the rhs of (2.16) the term

$$\pi \sum_{k} \operatorname*{Res}_{z=z_{f,k}} \frac{e^{i\pi z\sigma(z_{f,k})}}{\sin \pi z} f(z), \quad (2.20)$$

and to the rhs of (2.19) the term

$$-i\pi \sum_{k} \sigma(z_{f,k}) \operatorname*{Res}_{z=z_{f,k}} \frac{e^{i\pi z\sigma(z_{f,k})}}{\cos \pi z} f(z), \quad (2.21)$$

these formulae can be generalized to functions $f(z)$ that have non-real poles $z_{f,k}$, $\operatorname{Re} z_{f,k} > 0$. The second generalization with an application to the problem of diffraction scattering of charged particles is given in [61]. Another generalization of the APF, given in [62], is obtained from (2.11) taking in this formula $g(z) = -if(z)\cot \pi(z-\beta)$, $0 < \beta < 1$, with an analytic function $f(z)$.

As a next consequence of (2.11), a summation formula can be obtained over the points z_k, $\operatorname{Re} z_k > 0$ at which the analytic function $s(z)$ takes integer values, $s(z_k)$ is an integer, and $s'(z_k) \neq 0$. Taking in (2.11) $g(z) = -if(z)\cot \pi s(z)$, one obtains the following formula [63]

$$\sum_{k} \frac{f(z_k)}{s'(z_k)} = w + \int_{0}^{\infty} f(x)dx + \int_{0}^{\infty} dx \left[\frac{f(ix)}{e^{-2\pi i s(ix)} - 1} - \frac{f(-ix)}{e^{2\pi i s(-ix)} - 1}\right], \quad (2.22)$$

where

$$w = \begin{cases} 0, & \text{if} \quad s(0) \neq 0, \pm 1, \pm 2, \ldots \\ f(0)/[2s'(0)], & \text{if} \quad s(0) = 0, \pm 1, \pm 2, \ldots \end{cases} . \tag{2.23}$$

For $s(z) = z$ we return to the APF in the usual form. An example for the application of this formula to the Casimir effect is given in [63].

3 Applications to Bessel functions

Formula (2.11) contains two meromorphic functions and is too general. In order to obtain more special consequences we have to specify one of them. As we have seen in the previous section, one of possible ways leads to the APF. Here we will consider another choices of the function $g(z)$ and will obtain useful formulae for the series over zeros of Bessel functions and their combinations, as well as formulae for integrals involving these functions.

First of all, to simplify the formulae let us introduce the notation

$$\bar{F}(z) \equiv AF(z) + BzF'(z) \tag{3.1}$$

for a given function $F(z)$, where the prime denotes the derivative with respect to the argument of the function, A and B are real constants. As a function $g(z)$ in the GAPF let us choose

$$g(z) = i\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} f(z), \tag{3.2}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second (Neumann function) kind. For the sum and difference on the right of (2.11) one obtains

$$f(z) - (-1)^k g(z) = \frac{\bar{H}_\nu^{(k)}(z)}{\bar{J}_\nu(z)} f(z), \quad k = 1, 2, \tag{3.3}$$

with $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ being Bessel functions of the third kind or Hankel functions. For such a choice the integrals (2.1) and (2.10) can be estimated by using the asymptotic formulae for Bessel functions for fixed ν and $|z| \to \infty$ (see, for example, [64, 65]). It is easy to see that conditions (2.1) and (2.10) are satisfied if the function $f(z)$ is restricted to one of the following constraints

$$|f(z)| < \varepsilon(x) e^{c|y|} \quad \text{or} \quad |f(z)| < \frac{M e^{2|y|}}{|z|^\alpha}, \quad z = x + iy, \quad |z| \to \infty, \tag{3.4}$$

where $c < 2$, $\alpha > 1$ and $\varepsilon(x) \to 0$ for $x \to \infty$. Indeed, from asymptotic expressions for Bessel functions it follows that

$$\left| \int_{a\pm ih}^{b\pm ih} dz\, [g(z) \pm f(z)] \right| = \left| \int_a^b dx\, \frac{\bar{H}_\nu^{(j)}(x \pm ih)}{\bar{J}_\nu(x \pm ih)} f(x \pm ih) \right| < \begin{cases} M_1 e^{(c-2)h} \\ M_1'/h^\alpha \end{cases} \tag{3.5}$$

$$\left| \int_b^{b\pm i\infty} dz\, [g(z) \pm f(z)] \right| = \left| \int_0^\infty dx\, \frac{\bar{H}_\nu^{(j)}(b \pm ix)}{\bar{J}_\nu(b \pm ix)} f(b \pm ix) \right| < \begin{cases} N_1 \varepsilon(b) \\ N_1'/b^{\alpha-1} \end{cases} \tag{3.6}$$

with constants M_1, M_1', N_1, N_1', and $j = 1$ ($j = 2$) corresponds to the upper (lower) sign.

Let us denote by $\lambda_{\nu,k} \neq 0$, $k = 1, 2, 3 \ldots$, the zeros of $\bar{J}_\nu(z)$ in the right half-plane, arranged in ascending order of the real part, $\operatorname{Re}\lambda_{\nu,k} \leqslant \operatorname{Re}\lambda_{\nu,k+1}$, (if some of these zeros lie on the imaginary axis we will take only zeros with positive imaginary part). All these zeros are simple. Note that

for real $\nu > -1$ the function $\bar{J}_\nu(z)$ has only real zeros, except for the case $A/B + \nu < 0$ when there are two purely imaginary zeros [64]. By using the Wronskian $W[J_\nu(z), Y_\nu(z)] = 2/\pi z$, for (2.3) one finds

$$R[f(z), g(z)] = 2\sum_k T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)], \tag{3.7}$$

where we have introduced the notations

$$T_\nu(z) = \frac{z}{(z^2-\nu^2)\, \bar{J}_\nu^2(z) + z^2 J'^2_\nu(z)}\,, \tag{3.8}$$

$$\begin{aligned} r_{1\nu}[f(z)] &= \pi i \sum_k \underset{\mathrm{Im} z_k>0}{\mathrm{Res}} f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} - \pi i \sum_k \underset{\mathrm{Im} z_k<0}{\mathrm{Res}} f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)} - \\ &\quad -\pi \sum_k \underset{\mathrm{Im} z_k=0}{\mathrm{Res}} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}\,. \end{aligned} \tag{3.9}$$

Here z_k ($\neq \lambda_{\nu,i}$) are the poles of the function $f(z)$ in the region $\mathrm{Re}\, z > a > 0$. Substituting (3.7) into (2.11), we obtain that for the function $f(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant a$ and satisfying condition (3.4), the following formula takes place

$$\begin{aligned} &\lim_{b\to+\infty} \left\{ 2\sum_{k=m}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \mathrm{p.v.}\int_a^b dx\, f(x) \right\} = \\ &= -\frac{1}{2}\int_a^{a+i\infty} dz\, f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} - \frac{1}{2}\int_a^{a-i\infty} dz\, f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)}, \end{aligned} \tag{3.10}$$

where $\mathrm{Re}\, \lambda_{\nu,m-1} < a < \mathrm{Re}\, \lambda_{\nu,m}$, $\mathrm{Re}\, \lambda_{\nu,n} < b < \mathrm{Re}\, \lambda_{\nu,n+1}$, $a < \mathrm{Re}\, z_k < b$. We will apply this formula to the function $f(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant 0$ taking $a \to 0$. Let us consider two cases separately.

3.1 Case (a)

Let $f(z)$ has no poles on the imaginary axis, except possibly at $z = 0$, and

$$f(ze^{\pi i}) = -e^{2\nu\pi i} f(z) + o(z^{\beta_\nu}), \quad z \to 0 \tag{3.11}$$

(this condition is trivially satisfied for the function $f(z) = o(z^{\beta_\nu})$), with

$$\beta_\nu = \begin{cases} 2|\mathrm{Re}\,\nu| - 1, & \text{for integer } \nu, \\ \mathrm{Re}\,\nu + |\mathrm{Re}\,\nu| - 1, & \text{for non-integer } \nu. \end{cases} \tag{3.12}$$

Under this condition, for values ν, for which $\bar{J}_\nu(z)$ has no purely imaginary zeros, the rhs of Eq. (3.10) in the limit $a \to 0$ can be presented in the form

$$\begin{aligned} &-\frac{1}{\pi}\int_\rho^\infty dx\, \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f(xe^{\pi i/2}) + e^{\nu\pi i} f(xe^{-\pi i/2}) \right] \\ &+ \int_{\gamma_\rho^+} dz\, f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} - \int_{\gamma_\rho^-} dz\, f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)}, \end{aligned} \tag{3.13}$$

with γ_ρ^+ and γ_ρ^- being upper and lower halves of the semicircle in the right half-plane with radius ρ and with center at point $z = 0$, described in the positive sense with respect to this point. In

(3.13) we have introduced modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ [65]. It follows from (3.11) that for $z \to 0$

$$\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} f(z) = \frac{\bar{H}_\nu^{(2)}(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})} f(ze^{-\pi i}) + o(z^{-1}). \tag{3.14}$$

From here for $\rho \to 0$ one finds

$$D_\nu \equiv \int_{\gamma_\rho^+} dz\, f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} - \int_{\gamma_\rho^-} dz\, f(z) \frac{\bar{H}_\nu^{(2)}(z)}{\bar{J}_\nu(z)} = -\pi \operatorname*{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}. \tag{3.15}$$

Indeed,

$$\begin{aligned} D_\nu &= \int_{\gamma_\rho^+} dz\, f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} + \int_{\gamma_{1\rho}^+} dz\, f(ze^{-\pi i}) \frac{\bar{H}_\nu^{(2)}(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})} \\ &= \int_{\gamma_\rho^+ + \gamma_{1\rho}^+} dz\, f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} + \int_{\gamma_{1\rho}^+} dz\, o(z^{-1}) \\ &= i \int_{\gamma_\rho^+ + \gamma_{1\rho}^+} dz\, f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} + \int_{\gamma_{1\rho}^+} dz\, o(z^{-1}), \end{aligned} \tag{3.16}$$

where $\gamma_{1\rho}^+$ ($\gamma_{1\rho}^-$, see below) is the upper (lower) half of the semicircle with radius ρ in the left half-plane with the center at $z = 0$ (described in the positive sense). In the last equality we have used the condition that integral p.v.$\int_0^b dx\, f(x)$ converges at lower limit. In a similar way it can be seen that

$$D_\nu = i \int_{\gamma_\rho^- + \gamma_{1\rho}^-} dz\, f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} + \int_{\gamma_{1\rho}^-} dz\, o(z^{-1}). \tag{3.17}$$

Combining the last two results we obtain (3.15) in the limit $\rho \to 0$. By using (3.10), (3.13) and (3.15) we have [27]:

Theorem 2. *If f(z) is a single valued analytic function in the half-plane* $\operatorname{Re} z \geqslant 0$ *(with possible branch point at* $z = 0$*) except the poles* z_k $(\neq \lambda_{\nu,i})$, $\operatorname{Re} z_k > 0$ *(for the case of function* $f(z)$ *having purely imaginary poles see Remark after Theorem 3), and satisfy conditions (3.4) and (3.11), then in the case of* ν *for which the function* $\bar{J}_\nu(z)$ *has no purely imaginary zeros, the following formula is valid*

$$\begin{aligned} &\lim_{b \to +\infty} \left\{ 2 \sum_{k=1}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.} \int_0^b f(x) dx \right\} = \\ &= \frac{\pi}{2} \operatorname*{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} - \frac{1}{\pi} \int_0^\infty dx \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f(xe^{\frac{\pi i}{2}}) + e^{\nu\pi i} f(xe^{-\frac{\pi i}{2}}) \right], \end{aligned} \tag{3.18}$$

where on the left $\operatorname{Re} \lambda_{\nu,n} < b < \operatorname{Re} \lambda_{\nu,n+1}$, $0 < \operatorname{Re} z_k < b$, *and* $T_\nu(\lambda_{\nu,k})$ *and* $r_{1\nu}[f(z)]$ *are determined by relations (3.8) and (3.9).*

Under the condition (3.11) the integral on the right converges at lower limit. Recall that we assume the existence of the integral on the left as well (see section 2). Formula (3.18) and analogous ones given below are especially useful for numerical calculations of the sums over $\lambda_{\nu,k}$, as under the first conditions in (3.4) the integral on the right converges exponentially fast at the upper limit. Below, in Section 13 we will see that summation formula (3.18) may be generalized for the case when the coefficients in (3.1) are functions of z.

Remark. Deriving formula (3.18) we have assumed that the function $f(z)$ is meromorphic in the half-plane $\operatorname{Re} z \geqslant 0$ (except possibly at $z=0$). However this formula is valid also for some functions having branch points on the imaginary axis, for example,

$$f(z) = f_1(z) \prod_{l=1}^{k} \left(z^2 + c_l^2\right)^{\pm 1/2}, \tag{3.19}$$

with meromorphic function $f_1(z)$. The proof for (3.18) in this case is similar to the one given above with the difference that branch points $\pm i c_l$ have to be circled on the right along contours with small radii. In view of further applications to the Casimir effect let us consider the case $k=1$. By taking into account that

$$\left(x^2 e^{\pm \pi i} + c^2\right)^{1/2} = \begin{cases} \sqrt{c^2 - x^2}, & \text{if} \quad 0 \leqslant x \leqslant c, \\ e^{\pm i\pi/2}\sqrt{x^2 - c^2}, & \text{if} \quad x \geqslant c, \end{cases} \tag{3.20}$$

from (3.18) one obtains

$$\begin{aligned} \lim_{b\to+\infty} \left\{ 2\sum_{k=1}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.}\int_0^b dx\, f(x) \right\} &= \frac{\pi}{2} \operatorname*{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} \\ -\frac{1}{\pi}\int_0^c dx\, \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f_1(xe^{\pi i/2}) + e^{\nu\pi i} f_1(xe^{-\pi i/2}) \right] \left(c^2 - x^2\right)^{\pm 1/2} \\ \mp \frac{i}{\pi}\int_c^\infty dx\, \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f_1(xe^{\pi i/2}) - e^{\nu\pi i} f_1(xe^{-\pi i/2}) \right] \left(x^2 - c^2\right)^{\pm 1/2}, \end{aligned} \tag{3.21}$$

where $f(z) = f_1(z)\left(z^2 + c^2\right)^{\pm 1/2}$, $c > 0$. In Section 12 we apply this formula with an analytic function $f_1(z)$ to derive the Wightman function and the vacuum expectation values of the energy-momentum tensor in geometries with spherical boundaries. ■

For an analytic function $f(z)$ formula (3.18) yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2\lambda_{\nu,k} f(\lambda_{\nu,k})}{\left(\lambda_{\nu,k}^2 - \nu^2\right) J_\nu^2(\lambda_{\nu,k}) + \lambda_{\nu,k}^2 J_\nu'^2(\lambda_{\nu,k})} &= \int_0^\infty dx\, f(x) + \frac{\pi}{2}\operatorname*{Res}_{z=0} f(z) \frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)} \\ -\frac{1}{\pi}\int_0^\infty dx\, \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} \left[e^{-\nu\pi i} f(xe^{\pi i/2}) + e^{\nu\pi i} f(xe^{-\pi i/2}) \right]. \end{aligned} \tag{3.22}$$

By taking in this formula $\nu = 1/2$, $A=1$, $B=0$ (see notation (3.1)), as a particular case we immediately receive the APF in the form (2.16). In like manner, substituting $\nu = 1/2$, $A=1$, $B=2$, we obtain APF in the form (2.19). Consequently, the formula (3.18) is a generalization of the APF for general ν (with restrictions given above) and for functions $f(z)$ having poles in the right half-plane.

For further applications to the Casimir effect in Sections 12 and 14, let us choose in (3.22)

$$f(z) = F(z) J_{\nu+m}^2(zt), \quad t > 0, \quad \operatorname{Re}\nu \geqslant 0, \tag{3.23}$$

with m being an integer. Now the conditions (3.4) formulated in terms of $F(z)$ are of the form

$$|F(z)| < |z| \varepsilon e^{(c-2t)|y|} \quad \text{or} \quad |F(z)| < \frac{M e^{2(1-t)|y|}}{|z|^{\alpha-1}}, \; z = x + iy, \; |z| \to \infty, \tag{3.24}$$

with the same notations as in (3.4). In like manner, from condition (3.11) for $F(z)$ one has

$$F(ze^{\pi i}) = -F(z) + o(z^{-2m-1}), \quad z \to 0. \tag{3.25}$$

Now, as a consequence of (3.22) we obtain that if the conditions (3.24) and (3.25) are satisfied, then for a function $F(z)$ analytic in the right half-plane, the following formula takes place:

$$2\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) F(\lambda_{\nu,k}) J^2_{\nu+m}(\lambda_{\nu,k} t) = \int_0^\infty dx\, F(x) J^2_{\nu+m}(xt)$$
$$-\frac{1}{\pi}\int_0^\infty dx\, \frac{\bar{K}_\nu(x)}{\bar{I}_\nu(x)} I^2_{\nu+m}(xt) \left[F(xe^{\pi i/2}) + F(xe^{-\pi i/2})\right], \tag{3.26}$$

for $\mathrm{Re}\,\nu \geqslant 0$ and $\mathrm{Re}\,\nu + m \geqslant 0$.

3.2 Case (b)

Let $f(z)$ be a function satisfying the condition

$$f(xe^{\pi i/2}) = -e^{2\nu\pi i} f(xe^{-\pi i/2}), \tag{3.27}$$

for real x. It is clear that if $f(z)$ has purely imaginary poles, then they are complex conjugate: $\pm iy_k$, $y_k > 0$. By (3.27) the rhs of (3.10) for $a \to 0$ and $\arg \lambda_{\nu,k} = \pi/2$ may be written as

$$\sum_{\alpha=+,-} \alpha \left(\int_{\gamma_\rho^\alpha} + \sum_{\sigma_k = \alpha i y_k, \alpha\lambda_{\nu,k}} \int_{C_\rho(\sigma_k)} \right) \frac{\bar{H}_\nu^{(p_\alpha)}(z)}{\bar{J}_\nu(z)} f(z) dz, \tag{3.28}$$

where $p_+ = 1$, $p_- = 2$, $C_\rho(\sigma_k)$ denotes the right half of the circle with radius ρ and with the center at the point σ_k, described in the positive sense, and the contours $\gamma_\rho^\pm$ are the same as in (3.13). We have used the fact the purely imaginary zeros of $\bar{J}_\nu(z)$ are complex conjugate numbers, as $\bar{J}_\nu(ze^{\pi i}) = e^{\nu\pi i}\bar{J}_\nu(z)$. We have also used the fact that on the right of (3.10) the integrals (with $a = 0$) along straight segments of the upper and lower imaginary semiaxes are cancelled, as in accordance of (3.27) for $\arg z = \pi/2$

$$\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} f(z) = \frac{\bar{H}_\nu^{(2)}(ze^{-\pi i})}{\bar{J}_\nu(ze^{-\pi i})} f(ze^{-\pi i}). \tag{3.29}$$

Let us show that from (3.29) for $z_0 = x_0 e^{\pi i/2}$ it follows that this relation is valid for any z in a small enough region including this point. Namely, as the function $f(z)\bar{H}_\nu^{(p)}(z)/\bar{J}_\nu(z)$, $p = 1, 2$ is meromorphic near the point $(-1)^{p+1} x_0 e^{\pi i/2}$, there exists a neighborhood of this point where this function is presented as a Laurent expansion

$$\frac{\bar{H}_\nu^{(p)}(z)}{\bar{J}_\nu(z)} f(z) = \sum_{n=-n_0}^{\infty} \frac{a_n^{(p)}}{\left[z - (-1)^{p+1} x_0 e^{\pi i/2}\right]^n}. \tag{3.30}$$

From (3.29) for $z = xe^{\pi i/2}$ one concludes

$$\sum_{n=-n_0}^{\infty} \frac{a_n^{(1)} e^{-n\pi i/2}}{(x - x_0)^n} = \sum_{n=-n_0}^{\infty} \frac{(-1)^n a_n^{(2)} e^{-n\pi i/2}}{(x - x_0)^n}, \tag{3.31}$$

and hence $a_n^{(1)} = (-1)^n a_n^{(2)}$. Our statement follows directly from here. By this it can be seen that

$$\sum_{\alpha=+,-} \alpha \int_{C_\rho(\alpha\sigma_k)} dz \, \frac{\bar{H}_\nu^{(p_\alpha)}(z)}{\bar{J}_\nu(z)} f(z) = 2\pi i \operatorname*{Res}_{z=\sigma_k} \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)} f(z), \tag{3.32}$$

where $\sigma_k = iy_k$, $\lambda_{\nu,k}$, $\arg \lambda_{\nu,k} = \pi/2$. Now, by taking into account (3.15) and letting $\rho \to 0$ we get [27, 28]:

Theorem 3. *Let $f(z)$ be a meromorphic function in the half-plane $\operatorname{Re} z \geqslant 0$ (except possibly at $z = 0$) with poles z_k, $\operatorname{Re} z_k > 0$ and $\pm iy_k$, $y_k > 0$, $k = 1, 2, ...$ $(\neq \lambda_{\nu,p})$. If this function satisfies conditions (3.4) and (3.27) then*

$$\lim_{b\to+\infty} \left\{ 2\sum_{k=1}^{n} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}) + r_{1\nu}[f(z)] - \text{p.v.}\int_0^b f(x)dx \right\} =$$

$$= -\frac{\pi i}{2} \sum_{\eta_k=0,iy_k} (2 - \delta_{0\eta_k}) \operatorname*{Res}_{z=\eta_k} f(z) \frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}, \tag{3.33}$$

where on the left $0 < \operatorname{Re} z_k < b$, $\operatorname{Re} \lambda_{\nu,n} < b < \operatorname{Re} \lambda_{\nu,n+1}$ and $r_{1\nu}[f(z)]$ is defined by (3.9).

Note that the residue terms in (3.32) with $\sigma_k = \lambda_{\nu,k}$, $\arg \lambda_{\nu,k} = \pi/2$, are equal to $4T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k})$ and are included in the first sum on the left of (3.33).

Remark. Let $\pm iy_k$, $y_k > 0$, and $\pm\lambda_{\nu,k}$, $\arg \lambda_{\nu,k} = \pi/2$, be purely imaginary poles of function $f(z)$ and purely imaginary zeros of $\bar{J}_\nu(z)$, correspondingly. Let function $f(z)$ satisfy condition

$$f(z) = -e^{2\nu\pi i} f(ze^{-\pi i}) + o\left((z - \sigma_k)^{-1}\right), \quad z \to \sigma_k, \quad \sigma_k = iy_k, \lambda_{\nu,k}. \tag{3.34}$$

Now in the limit $a \to 0$ the rhs of (3.10) can be presented in the form (3.28) plus integrals along the straight segments of the imaginary axis between the poles. Using the arguments similar those given above, we obtain relation (3.32) with an additional contribution from the last term on the right of (3.34) in the form $\int_{C_\rho(-\sigma_k)} o\left((z - \sigma_k)^{-1}\right) dz$. In the limit $\rho \to 0$ the latter vanishes and the sum of the integrals along the straight segments of the imaginary axis gives the principal value of the integral on the right of (3.18). As a result formula (3.18) can be generalized for functions having purely imaginary poles and satisfying condition (3.34). For this on the right of (3.18) instead of residue term we have to write the sum of residues from the rhs of (3.33) and take the principal value of the integral on the right. The latter exists due to condition (3.34). ■

An interesting result can be obtained from (3.33). Let $\lambda_{\mu,k}^{(1)}$ be zeros of the function $A_1 J_\mu(z) + B_1 z J'_\mu(z)$ with some real constants A_1 and B_1. Let $f(z)$ be an analytic function in the right half-plane satisfying condition (3.27) and $f(z) = o(z^\beta)$ for $z \to 0$, where $\beta = \max(\beta_\mu, \beta_\nu)$ (the definition β_ν see (3.12)). For this function we get from (3.33):

$$\sum_{k=1}^{\infty} T_\mu(\lambda_{\mu,k}^{(1)}) f(\lambda_{\mu,k}^{(1)}) = \sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) f(\lambda_{\nu,k}), \quad \mu = \nu + m\,. \tag{3.35}$$

For the case of Fourier-Bessel and Dini series this result is given in [64].

Let us consider some applications of formula (3.33) to the special types of series. First we choose in this formula

$$f(z) = F_1(z) J_\mu(zt), \quad t > 0, \tag{3.36}$$

where the function $F_1(z)$ is meromorphic on the right half-plane and satisfies the conditions

$$|F_1(z)| < \varepsilon_1(x)e^{(c-t)|y|} \quad \text{or} \quad |F_1(z)| < M|z|^{-\alpha_1}e^{(2-t)|y|},\ |z| \to \infty, \tag{3.37}$$

with $c < 2$, $\alpha_1 > 1/2$, $\varepsilon_1(x) = o(\sqrt{x})$ for $x \to +\infty$, and the condition

$$F_1(xe^{\pi i/2}) = -e^{(2\nu-\mu)\pi i}F_1(xe^{-\pi i/2}). \tag{3.38}$$

From (3.37) it follows that the integral p.v.$\int_0^\infty dx\, F_1(x)J_\mu(xt)$ converges at the upper limit and, hence, in this case formula (3.33) may be written in the form

$$\begin{aligned}\sum_{k=1}^\infty T_\nu(\lambda_{\nu,k})F_1(\lambda_{\nu,k})J_\mu(\lambda_{\nu,k}t) &= \frac{1}{2}\text{p.v.}\int_0^\infty dx\, F_1(x)J_\mu(xt) - \frac{1}{2}r_{1\nu}\left[F_1(z)J_\mu(zt)\right] \\ &-\frac{\pi i}{4}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\operatorname*{Res}_{z=\eta_k}F_1(z)J_\mu(zt)\frac{\bar{H}_\nu^{(1)}(z)}{\bar{J}_\nu(z)}.\end{aligned} \tag{3.39}$$

Note that the lhs of this formula is known as Dini series of Bessel functions [64]. For example, it follows from here that for $t < 1$, $\operatorname{Re}\sigma$, $\operatorname{Re}\nu > -1$ one has

$$\begin{aligned}\sum_{k=1}^\infty \frac{T_\nu(\lambda_{\nu,k})}{\lambda_{\nu,k}^\sigma}J_{\sigma+\nu+1}(\lambda_{\nu,k})J_\nu(\lambda_{\nu,k}t) &= \frac{1}{2}\int_0^\infty \frac{dz}{z^\sigma}J_{\sigma+\nu+1}(z)J_\nu(zt) \\ &= \frac{\left(1-t^2\right)^\sigma t^\nu}{2^{\sigma+1}\Gamma(\sigma+1)},\end{aligned} \tag{3.40}$$

(for the value of the integral see, e.g., [64]). For $B = 0$ this result is given in [66]. In a similar way, taking $\mu = \nu + m$,

$$F_1(z) = z^{\nu+m+1}\frac{J_\sigma(a\sqrt{z^2+z_1^2})}{\left(z^2+z_1^2\right)^{\sigma/2}}, \quad a > 0, \tag{3.41}$$

with $\operatorname{Re}\nu \geqslant 0$ and $\operatorname{Re}\nu + m \geqslant 0$, from (3.39) for $a < 2 - t$, $\operatorname{Re}\sigma > \operatorname{Re}\nu + m$ one finds

$$\begin{aligned}&\sum_{k=1}^\infty T_\nu(\lambda_{\nu,k})\lambda_{\nu,k}^{\nu+m+1}J_{\nu+m}(\lambda_{\nu,k}t)\frac{J_\sigma(a\sqrt{\lambda_{\nu,k}^2+z_1^2})}{\left(\lambda_{\nu,k}^2+z_1^2\right)^{\sigma/2}} = \\ &= \frac{1}{2}\int_0^\infty dx\, x^{\nu+m+1}J_{\nu+m}(xt)\frac{J_\sigma(a\sqrt{x^2+z_1^2})}{\left(x^2+z_1^2\right)^{\sigma/2}} \\ &= \frac{t^{\nu+1}}{a^\sigma}\left(-z_1\right)^{m+1}\frac{J_{m+1}(z_1\sqrt{a^2-t^2})}{\left(a^2-t^2\right)^{(m+1)/2}},\ a > t,\end{aligned} \tag{3.42}$$

and the sum is zero when $a < t$. Here we have used the known value for the Sonine integral [64].

If an addition to (3.37), (3.38) the function $F_1(z)$ satisfies the conditions

$$F_1(xe^{\pi i/2}) = -e^{\mu\pi i}F_1(xe^{-\pi i/2}) \tag{3.43}$$

and

$$|F_1(z)| < \varepsilon_1(x)e^{c_1 t|y|} \quad \text{or} \quad |F_1(z)| < M|z|^{-\alpha_1}e^{t|y|}, \quad |z| \to \infty, \tag{3.44}$$

then formula (6.7) (see below) with $B = 0$ may be applied to the integral on the right of (3.39). This gives [27, 28]:

Corollary 1. *Let $F(z)$ be a meromorphic function in the half-plane* $\operatorname{Re} z \geqslant 0$ *(except possibly at $z = 0$) with poles z_k,* $\operatorname{Re} z_k > 0$ *and $\pm iy_k$, $y_k > 0$ ($\neq \lambda_{\nu,i}$). If $F(z)$ satisfy condition*

$$F(xe^{\pi i/2}) = (-1)^{m+1} e^{\nu\pi i} F(xe^{-\pi i/2}), \tag{3.45}$$

with an integer m, and one of the inequalities

$$|F(z)| < \varepsilon_1(x) e^{a|y|} \quad \text{or} \quad |F(z)| < M|z|^{-\alpha_1} e^{a_0|y|}, \quad |z| \to \infty, \tag{3.46}$$

with $a < \min(t, 2-t) \equiv a_0$, $\varepsilon_1(x) = o(x^{1/2})$, $x \to +\infty$, $\alpha_1 > 1/2$, the following formula is valid

$$\begin{aligned}\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) F(\lambda_{\nu,k}) J_{\nu+m}(\lambda_{\nu,k} t) &= \frac{\pi i}{4} \sum_{\eta_k = 0, iy_k, z_k} (2 - \delta_{0\eta_k}) \\ \times \operatorname*{Res}_{z=\eta_k} &\left\{ \left[J_{\nu+m}(zt) \bar{Y}_\nu(z) - Y_{\nu+m}(zt) \bar{J}_\nu(z) \right] \frac{F(z)}{\bar{J}_\nu(z)} \right\}.\end{aligned} \tag{3.47}$$

Recall that for the imaginary zeros $\lambda_{\nu,k}$, on the lhs of (3.47) the zeros with positive imaginary parts enter only. By using formula (3.47) a number of Fourier-Bessel and Dini series can be summarized (see below).

Remark. Formula (3.47) may also be obtained by considering the integral

$$\frac{1}{\pi} \int_{C_h} dz \left[H_{\nu+m}^{(2)}(zt) \bar{H}_\nu^{(1)}(z) - H_{\nu+m}^{(1)}(zt) \bar{H}_\nu^{(2)}(z) \right] \frac{F(z)}{\bar{J}_\nu(z)}, \tag{3.48}$$

where C_h is a rectangle with vertices $\pm ih$, $b \pm ih$, described in the positive sense (purely imaginary poles of $F(z)/\bar{J}_\nu(z)$ and the origin are circled by semicircles in the right half-plane with small radii). This integral is equal to the sum of residues over the poles within C_h (points z_k, $\lambda_{\nu,k}$, ($\operatorname{Re} z_k$, $\operatorname{Re} \lambda_{\nu,k} > 0$)). On the other hand, it follows from (3.45) that the integrals along the segments of the imaginary axis cancel each other. The sum of integrals along the conjugate semicircles give the sum of residues over purely imaginary poles in the upper half-plane. The integrals along the remaining three segments of C_h in accordance with (3.46) approach to zero in the limit b, $h \to \infty$. Equating these expressions for (3.48), one immediately obtains the result (3.47). ■

From (3.47), for $t = 1$, $F(z) = J_\nu(zx)$, $m = 1$, one obtains [13, 64]

$$\sum_{k=1}^{\infty} T_\nu(\lambda_{\nu,k}) J_\nu(\lambda_{\nu,k} x) J_{\nu+1}(\lambda_{\nu,k}) = \frac{x^\nu}{2}, \quad 0 \leqslant x < 1. \tag{3.49}$$

In a similar way, choosing $m = 0$, $F(z) = zJ_\nu(zx)/\left(z^2 - a^2\right)$, $B = 0$, we obtain the Kneser-Sommerfeld expansion [64]:

$$\sum_{k=1}^{\infty} \frac{J_\nu(\lambda_{\nu,k} t) J_\nu(\lambda_{\nu,k} x)}{\left(\lambda_{\nu,k}^2 - a^2\right) J_{\nu+1}^2(\lambda_{\nu,k})} = \frac{\pi}{4} \frac{J_\nu(ax)}{J_\nu(a)} \left[J_\nu(at) Y_\nu(a) - Y_\nu(at) J_\nu(a) \right], \tag{3.50}$$

for $0 \leqslant x \leqslant t \leqslant 1$. In (3.47) as a function $F(z)$ one may choose, for example, the following functions

$$z^{\rho-1}\prod_{l=1}^{n}\left(z^2+z_l^2\right)^{-\mu_l/2}J_{\mu_l}(b_l\sqrt{z^2+z_l^2}),\quad b\leqslant a_0,\, b=\sum_{l=1}^{n}b_l,$$

$$\mathrm{Re}\,\nu<\sum_{l=1}^{n}\mathrm{Re}\,\mu_l+n/2+2p+3/2-m-\delta_{ba_0}; \tag{3.51}$$

$$z^{\rho-2n-1}\prod_{l=1}^{n}\left[1-J_0(b_lz)\right],\quad \mathrm{Re}\,\nu<2n+2p+3/2-m-\delta_{ba_0}; \tag{3.52}$$

$$z^{\rho-1}\prod_{l=1}^{n}\left(z^2+z_l^2\right)^{\mu_l/2}Y_{\mu_l}\left(b_l\sqrt{z^2+z_l^2}\right),\ \mu_l>0\text{-half of an odd integer},$$

$$\mathrm{Re}\,\nu<-\sum_{l=1}^{n}\mu_l+n/2+2p+3/2-m-\delta_{ba_0}; \tag{3.53}$$

$$z^{\rho-1}\prod_{l=1}^{n}z^{|k_l|}\left[J_{\mu_l+k_l}(a_lz)Y_{\mu_l}(b_lz)-Y_{\mu_l+k_l}(a_lz)J_{\mu_l}(b_lz)\right],\quad k_l\text{ - integer},$$

$$\mathrm{Re}\,\nu<n+2p+3/2-m-\sum|k_l|-\delta_{\tilde{a},a_0},\ \tilde{a}\equiv\sum_{l=1}^{n}|a_l-b_l|\leqslant a_0; \tag{3.54}$$

with $\rho=\nu+m-2p$ (p is an integer), as well as any products between these functions and with $\prod_l\left(z^2-c_l^2\right)^{-p_l}$, provided condition (3.46) is satisfied. For example, the following formulae take place

$$\sum_{k=1}^{\infty}j_{\nu,k}^{\nu-2}\frac{J_\nu(j_{\nu,k}t)}{J_{\nu+1}^2(j_{\nu,k})}\prod_{l=1}^{n}\left[J_{\mu_l}(a_lj_{\nu,k})Y_{\mu_l}(b_lj_{\nu,k})-Y_{\mu_l}(a_lj_{\nu,k})J_{\mu_l}(b_lj_{\nu,k})\right]$$

$$=\frac{2^{\nu-2}}{\pi^nt^\nu}\left(1-t^{2\nu}\right)\prod_{l=1}^{n}\frac{b_l^{\mu_l}}{\mu_la_l^{\mu_l}}\left[1-\left(\frac{a_l}{b_l}\right)^{2\mu_l}\right],\quad 0<t\leqslant 1,$$

$$c\equiv\sum_{l=1}^{n}|a_l-b_l|\leqslant t,\quad a_l,b_l>0,\quad \mathrm{Re}\,\mu_l\geqslant 0,\quad \mathrm{Re}\,\nu<n+3/2-\delta_{ct}; \tag{3.55}$$

$$\sum_{k=1}^{\infty}\frac{J_\nu(j_{\nu,k}t)J_{\nu+1}(\lambda j_{\nu,k})}{j_{\nu,k}^{2n+3}J_{\nu+1}^2(j_{\nu,k})}\prod_{l=1}^{n}\left[1-J_0(b_lj_{\nu,k})\right]=\frac{\lambda^{\nu+1}\left(1-t^{2\nu}\right)}{4^{n+1}\nu(\nu+1)t^\nu}\prod_{l=1}^{n}b_l^2,$$

$$\lambda+\sum_{l=1}^{n}b_l\leqslant t\leqslant 1,\quad \lambda,b_l>0; \tag{3.56}$$

$$\sum_{k=1}^{\infty}\frac{J_\mu(j_{\nu,k}b)J_{\nu+1}(\lambda j_{\nu,k})J_\nu(j_{\nu,k}t)}{\left(j_{\nu,k}^2-a^2\right)j_{\nu,k}^{\mu+1}J_{\nu+1}^2(j_{\nu,k})}=\frac{\pi J_{\nu+1}(a\lambda)}{4a^{\mu+1}}\left[Y_\nu(a)J_\nu(at)-J_\nu(a)Y_\nu(at)\right],$$

$$\times\frac{J_\mu(ba)}{J_\nu(a)},\ \lambda+b\leqslant t\leqslant 1,\ \lambda,b>0,\ \mathrm{Re}\mu>-7/2+\delta_{\lambda+b,t}, \tag{3.57}$$

where $j_{\nu,k}$ are zeros of the function $J_\nu(z)$. The examples of the series over zeros of Bessel functions we found in literature (see, e.g., [13, 64, 66, 67]), when the corresponding sum was evaluated in finite terms, are particular cases of the formulae given in this section.

4 Summation formulae over zeros of combinations of cylinder functions

Here we will consider series over the zeros of the function

$$C_{\nu}^{ab}(\eta,z)\equiv \bar{J}_{\nu}^{(a)}(z)\bar{Y}_{\nu}^{(b)}(\eta z)-\bar{Y}_{\nu}^{(a)}(z)\bar{J}_{\nu}^{(b)}(\eta z), \tag{4.1}$$

where and in what follows for a given function $f(z)$ the quantities with overbars are defined by the formula

$$\bar{f}^{(j)}(z)=A_j f(x)+B_j z f'(z),\ j=a,b, \tag{4.2}$$

with constant coefficients A_j and B_j. This type of series arises in calculations of the vacuum expectation values in confined regions with boundaries of spherical and cylindrical form. To obtain a summation formula for these series, in the GAPF we substitute

$$g(z)=\frac{1}{2i}\left[\frac{\bar{H}_{\nu}^{(1b)}(\eta z)}{\bar{H}_{\nu}^{(1a)}(z)}+\frac{\bar{H}_{\nu}^{(2b)}(\eta z)}{\bar{H}_{\nu}^{(2a)}(z)}\right]\frac{h(z)}{C_{\nu}^{ab}(\eta,z)},\quad f(z)=\frac{h(z)}{\bar{H}_{\nu}^{(1a)}(z)\bar{H}_{\nu}^{(2a)}(z)}, \tag{4.3}$$

where for definiteness we shall assume that $\eta>1$. The sum and difference of these functions are given by the formula

$$g(z)-(-1)^k f(z)=-i\frac{\bar{H}_{\nu}^{(ka)}(\lambda z)}{\bar{H}_{\nu}^{(ka)}(z)}\frac{h(z)}{C_{\nu}^{ab}(\eta,z)},\quad k=1,2. \tag{4.4}$$

The condition for the GAPF written in terms of the function $h(z)$ takes the form

$$|h(z)|<\varepsilon_1(x)e^{c_1|y|}\quad |z|\to\infty,\quad z=x+iy, \tag{4.5}$$

where $c_1<2(\eta-1)$ and $x^{\delta_{B_a0}+\delta_{B_b0}-1}\varepsilon_1(x)\to 0$ for $x\to+\infty$. Let $\gamma_{\nu,k}$ be zeros of the function $C_{\nu}^{ab}(\eta,z)$ in the right half-plane. In this section we will assume values of ν, A_j, and B_j for which all these zeros are real and simple, and the function $\bar{H}_{\nu}^{(1a)}(z)$ ($\bar{H}_{\nu}^{(2a)}(z)$) has no zeros in the right half of the upper (lower) half-plane. For real ν and A_j, B_j the zeros $\gamma_{\nu,k}$ are simple. To see this note that the function $J_{\nu}(tz)\bar{Y}_{\nu}^{(a)}(z)-Y_{\nu}(tz)\bar{J}_{\nu}^{(a)}(z)$ is a cylinder function with respect to t. Using the standard result for indefinite integrals containing the product of any two cylinder functions (see [64, 65]), it can be seen that

$$\int_1^{\eta}dt\,t\left[J_{\nu}(tz)\bar{Y}_{\nu}^{(a)}(z)-Y_{\nu}(tz)\bar{J}_{\nu}^{(a)}(z)\right]^2=\frac{2}{\pi^2 z T_{\nu}^{ab}(\eta,z)},\quad z=\gamma_{\nu,k}, \tag{4.6}$$

where we have introduced the notation

$$T_{\nu}^{ab}(\eta,z)=z\left\{\frac{\bar{J}_{\nu}^{(a)2}(z)}{\bar{J}_{\nu}^{(b)2}(\eta z)}\left[A_b^2+B_b^2(\eta^2z^2-\nu^2)\right]-A_a^2-B_a^2(z^2-\nu^2)\right\}^{-1}. \tag{4.7}$$

On the other hand

$$\frac{\partial}{\partial z}C_{\nu}^{ab}(\eta,z)=\frac{2}{\pi T_{\nu}^{ab}(\eta,z)}\frac{\bar{J}_{\nu}^{(b)}(\eta z)}{\bar{J}_{\nu}^{(a)}(z)},\quad z=\gamma_{\nu,k}. \tag{4.8}$$

Combining the last two results we deduce that for real ν, A_j, B_j the derivative (4.8) is nonzero and, hence, the zeros $z=\gamma_{\nu,k}$ are simple. By using this we can see that

$$\underset{z=\gamma_{\nu,k}}{\mathrm{Res}}\,g(z)=\frac{\pi}{2i}T_{\nu}^{ab}(\eta,\gamma_{\nu,k})h(\gamma_{\nu,k}). \tag{4.9}$$

Hence, if the function $h(z)$ is analytic in the half-plane $\mathrm{Re}\, z \geqslant a > 0$ except at the poles z_k ($\neq \gamma_{\nu,i}$) and satisfies condition (4.5), the following formula takes place

$$\lim_{b\to+\infty}\left\{\frac{\pi^2}{2}\sum_{k=n}^{m} T_\nu^{ab}(\lambda,\gamma_{\nu,k})h(\gamma_{\nu,k}) + r_{2\nu}[h(z)] - \mathrm{p.v.}\int_a^b \frac{h(x)dx}{\bar{J}_\nu^{(a)2}(x)+\bar{Y}_\nu^{(a)2}(x)}\right\}$$
$$= \frac{i}{2}\int_a^{a+i\infty} \frac{\bar{H}_\nu^{(1b)}(\lambda z)}{\bar{H}_\nu^{(1a)}(z)}\frac{h(z)}{C_\nu^{ab}(\lambda,z)}dz - \frac{i}{2}\int_a^{a-i\infty}\frac{\bar{H}_\nu^{(2b)}(\lambda z)}{\bar{H}_\nu^{(2a)}(z)}\frac{h(z)}{C_\nu^{ab}(\lambda,z)}dz, \tag{4.10}$$

where we have assumed that the integral on the left exists. In formula (4.10), $\gamma_{\nu,n-1} < a < \gamma_{\nu,n}$, $\gamma_{\nu,m} < b < \gamma_{\nu,m+1}$, $a < \mathrm{Re}\, z_k < b$, $\mathrm{Re}\, z_k \leqslant \mathrm{Re}\, z_{k+1}$, and the following notation is introduced

$$\begin{aligned} r_{2\nu}[h(z)] &= \pi\sum_k \underset{\mathrm{Im} z_k=0}{\mathrm{Res}}\left[\frac{\bar{J}_\nu^{(a)}(z)\bar{J}_\nu^{(b)}(\lambda z)+\bar{Y}_\nu^{(a)}(z)\bar{Y}_\nu^{(b)}(\lambda z)}{\bar{J}_\nu^{(a)2}(z)+\bar{Y}_\nu^{(a)2}(z)}\frac{h(z)}{C_\nu^{ab}(\lambda,z)}\right] \\ &+\pi\sum_{k,l=1,2}\underset{(-1)^l \mathrm{Im} z_k<0}{\mathrm{Res}}\left[\frac{\bar{H}_\nu^{(lb)}(\lambda z)}{\bar{H}_\nu^{(la)}(z)}\frac{h(z)}{C_\nu^{ab}(\lambda,z)}\right]. \end{aligned} \tag{4.11}$$

General formula (4.10) is a direct consequence of the GAPF and will be used as a starting point for further applications in this section. In the limit $a \to 0$ one has [27, 28, 38]:

Corollary 2. *Let $h(z)$ be an analytic function for $\mathrm{Re}\, z \geqslant 0$ except the poles z_k ($\neq \gamma_{\nu i}$), $\mathrm{Re}\, z_k > 0$ (with possible branch point $z = 0$). If it satisfies condition (4.5) and*

$$h(ze^{\pi i}) = -h(z) + o(z^{-1}), \quad z \to 0, \tag{4.12}$$

and the integral

$$\mathrm{p.v.}\int_0^b \frac{h(x)dx}{\bar{J}_\nu^{(a)2}(x)+\bar{Y}_\nu^{(a)2}(x)} \tag{4.13}$$

exists, then

$$\lim_{b\to+\infty}\left\{\frac{\pi^2}{2}\sum_{k=1}^{m} h(\gamma_{\nu,k})T_\nu^{ab}(\lambda,\gamma_{\nu,k}) + r_{2\nu}[h(z)] - \mathrm{p.v.}\int_0^b \frac{h(x)dx}{\bar{J}_\nu^{(a)2}(x)+\bar{Y}_\nu^{(a)2}(x)}\right\}$$
$$= -\frac{\pi}{2}\underset{z=0}{\mathrm{Res}}\left[\frac{h(z)\bar{H}_\nu^{(1b)}(\lambda z)}{C_\nu^{ab}(\lambda,z)\bar{H}_\nu^{(1a)}(z)}\right] - \frac{\pi}{4}\int_0^\infty dx\, \Omega_{a\nu}(x,\lambda x)\left[h(xe^{\frac{\pi i}{2}})+h(xe^{-\frac{\pi i}{2}})\right]. \tag{4.14}$$

In formula (4.14) we have introduced the notation

$$\Omega_{a\nu}(x,\lambda x) = \frac{\bar{K}_\nu^{(b)}(\lambda x)/\bar{K}_\nu^{(a)}(x)}{\bar{K}_\nu^{(a)}(x)\bar{I}_\nu^{(b)}(\lambda x)-\bar{K}_\nu^{(b)}(\lambda x)\bar{I}_\nu^{(a)}(x)}. \tag{4.15}$$

In the following we shall use this formula to derive the renormalized vacuum energy-momentum tensor for the region between two spherical and cylindrical surfaces. Note that (4.14) may be generalized for functions $h(z)$ with purely imaginary poles $\pm iy_k$, $y_k > 0$, satisfying the condition

$$h(ze^{\pi i}) = -h(z) + o\left((z \mp iy_k)^{-1}\right), \quad z \to \pm iy_k. \tag{4.16}$$

The corresponding formula is obtained from (4.14) by adding residue terms for $z = iy_k$ in the form of (4.18) (see below) and taking the principal value of the integral on the right. The arguments here are similar to those for Remark after Theorem 3.

In a way similar to (3.18), one has another result [27, 28]:

Corollary 3. *Let $h(z)$ be a meromorphic function in the half-plane* $\operatorname{Re} z \geqslant 0$ *(with exception the possible branch point $z=0$) with poles z_k, $\pm iy_k$ ($\neq \gamma_{\nu,i}$),* $\operatorname{Re} z_k, y_k > 0$. *If this function satisfies the condition*

$$h(xe^{\pi i/2}) = -h(xe^{-\pi i/2}), \tag{4.17}$$

and the integral (4.13) exists, then

$$\lim_{b\to+\infty}\left\{\frac{\pi^2}{2}\sum_{k=1}^{m} h(\gamma_{\nu,k})T_\nu^{ab}(\lambda,\gamma_{\nu,k}) + r_{2\nu}[h(z)] - \text{p.v.}\int_0^b \frac{h(x)dx}{\bar{J}_\nu^{(a)2}(x)+\bar{Y}_\nu^{(a)2}(x)}\right\}$$
$$= -\frac{\pi}{2}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\operatorname*{Res}_{z=\eta_k}\left[\frac{\bar{H}_\nu^{(1b)}(\lambda z)}{\bar{H}_\nu^{(1a)}(z)}\frac{h(z)}{C_\nu^{ab}(\lambda,z)}\right], \tag{4.18}$$

where on the left $\gamma_{\nu,m} < b < \gamma_{\nu,m+1}$.

Let us consider a special application of formula (4.18) for $A_j = 1$, $B_j = 0$. Under the conditions given above, the generalizations of these results for general A_j, B_j are straightforward.

Theorem 4. *Let the function $F(z)$ be meromorphic in the right half-plane* $\operatorname{Re} z \geqslant 0$ *(with the possible exception $z=0$) with poles $z_k, \pm iy_k$ ($\neq \gamma_{\nu,i}$), $y_k, \operatorname{Re} z_k > 0$. If it satisfies condition*

$$F(xe^{\pi i/2}) = (-1)^{m+1}F(xe^{-\pi i/2}), \tag{4.19}$$

with an integer m, and one of two inequalities

$$|F(z)| < \varepsilon(x)e^{a_1|y|} \quad \text{or} \quad |F(z)| < M|z|^{-\alpha}e^{a_2|y|}, \quad |z|\to\infty, \tag{4.20}$$

with $a_1 < \min(2\lambda-\sigma-1, \sigma-1) \equiv a_2$, $\sigma > 0$, $\varepsilon(x)\to 0$ for $x\to+\infty$, $\alpha > 1$, then

$$\sum_{k=1}^{\infty}\frac{\gamma_{\nu,k}F(\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k})-1}\left[J_\nu(\gamma_{\nu,k})Y_{\nu+m}(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k})J_{\nu+m}(\sigma\gamma_{\nu,k})\right]$$
$$= \sum_{\eta_k=0,iy_k,z_k}\frac{2-\delta_{0\eta_k}}{\pi}\operatorname*{Res}_{z=\eta_k}\frac{Y_\nu(\lambda z)J_{\nu+m}(\sigma z) - J_\nu(\lambda z)Y_{\nu+m}(\sigma z)}{J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)}F(z). \tag{4.21}$$

Proof. In (4.18) let us choose

$$h(z) = F(z)\left[J_\nu(z)Y_{\nu+m}(\sigma z) - Y_\nu(z)J_{\nu+m}(\sigma z)\right], \tag{4.22}$$

which in virtue of (4.20) satisfies condition (4.5). Condition (4.17) is satisfied as well. Hence, $h(z)$ satisfies the conditions for Corollary 3. The corresponding integral in (4.18) with $h(z)$ from (4.22) can be evaluated by using formula (8.7) (see below). Putting the value of this integral into (4.18) after some manipulations we obtain formula (4.21). ■

Remark. Formula (4.21) may also be derived by applying to the contour integral

$$\int_{C_h}\frac{Y_\nu(\lambda z)J_{\nu+m}(\sigma z) - J_\nu(\lambda z)Y_{\nu+m}(\sigma z)}{J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)}F(z)dz \tag{4.23}$$

the residue theorem, where C_h is a rectangle with vertices $\pm ih$, $b\pm ih$. Here the proof is similar to that for Remark to Corollary 1. ■

By using (4.18) a formula similar to (4.21) can also be obtained for series of the type $\sum_{k=1}^{\infty}G(\gamma_{\nu,k})J_\mu(\gamma_{\nu,k}t)$.

As a function $F(z)$ in (4.21) one can choose, for example,

- function (3.51) for $\rho = m - 2p$, $\sum_l b_l < a_2$, $m < 2p + \sum_l \operatorname{Re}\mu_l + n/2 + 1$, p - integer;
- function (3.52) for $\rho = m - 2p$, $\sum_l b_l < a_2$, $m < 2p + 2n + 1$;
- function (3.54) for $\rho = m - 2p$, $a_l > 0$, $\operatorname{Re}\mu_l \geqslant 0$ (for $\operatorname{Re}\mu_l < 0$, $k_l > |\operatorname{Re}\mu_l|$), $\sum_{l=1}^{n} |a_l - b_l| < a_2$, $m < 2p + n - \sum_l |k_l| + 1$.

Taking $F(z) = 1/z$, $m = 0$, one obtains

$$\sum_{k=1}^{\infty} \frac{J_\nu(\gamma_{\nu,k}) Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k}) J_\nu(\sigma\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1} = \frac{\sigma^\nu}{\pi} \frac{(\lambda/\sigma)^{2\nu} - 1}{\lambda^{2\nu} - 1}, \quad \lambda \geqslant \sigma > 1. \tag{4.24}$$

In a similar way it can be seen that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\gamma_{\nu,k}^2 \left[J_\nu(\gamma_{\nu,k}) Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k}) J_\nu(\sigma\gamma_{\nu,k})\right]}{\left(\gamma_{\nu,k}^2 - c^2\right) \left[J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1\right]} = \\
&\quad = \frac{1}{\pi} \frac{Y_\nu(\lambda c) J_\nu(\sigma c) - J_\nu(\lambda c) Y_\nu(\sigma c)}{J_\nu(c) Y_\nu(\lambda c) - J_\nu(\lambda c) Y_\nu(c)},
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{J_\nu(\gamma_{\nu,k}) Y_\nu(\sigma\gamma_{\nu,k}) - Y_\nu(\gamma_{\nu,k}) J_\nu(\sigma\gamma_{\nu,k})}{J_\nu^2(\gamma_{\nu,k})/J_\nu^2(\lambda\gamma_{\nu,k}) - 1} \prod_{l=1}^{p} \frac{J_{\mu_l}(b_l \gamma_{\nu,k})}{\gamma_{\nu,k}^{\mu_l}} = \frac{\sigma^\nu}{\pi} \frac{(\lambda/\sigma)^{2\nu} - 1}{\lambda^{2\nu} - 1} \\
&\quad \times \prod_{l=1}^{p} \frac{2^{-\mu_l} b_l^{\mu_l}}{\Gamma(\mu_l + 1)}, \quad b \equiv \sum_{1}^{p} b_l < \sigma - 1, \ \operatorname{Re}\mu_l + \frac{p}{2} + 1 > \delta_{b,\sigma-1},
\end{aligned} \tag{4.26}$$

where $\operatorname{Re} c \geqslant 0$, $b_l > 0$, $\lambda \geqslant \sigma > 1$, $\mu_l \neq -1, -2, \ldots$. Physical applications of the formulae derived in this section will be considered below.

5 Summation formulae over the zeros of modified Bessel functions with an imaginary order

5.1 Summation formula over the zeros of the function $\bar{K}_{iz}(\eta)$

We denote by $z = \omega_n = \omega_n(\eta)$ the zeros of the function

$$\bar{K}_{iz}(\eta) \equiv A K_{iz}(\eta) + B\eta \partial_\eta K_{iz}(\eta) \tag{5.1}$$

in the half-plane $\operatorname{Re} z > 0$ for a given η:

$$\bar{K}_{i\omega_n}(\eta) = 0, \quad n = 1, 2, \ldots \tag{5.2}$$

A summation formula for the series over these zeros can be obtained by making use of the GAPF. For this, as functions $f(z)$ and $g(z)$ in formula (2.11) we choose

$$f(z) = \frac{2i}{\pi} F(z) \sinh \pi z, \quad g(z) = \frac{\bar{I}_{iz}(\eta) + \bar{I}_{-iz}(\eta)}{\bar{K}_{iz}(\eta)} F(z), \tag{5.3}$$

with a function $F(z)$ meromorphic in the right half-plane $\operatorname{Re} z > 0$. For the sum and difference of these functions one finds

$$g(z) \pm f(z) = 2F(z) \frac{\bar{I}_{\mp iz}(\eta)}{\bar{K}_{iz}(\eta)}. \tag{5.4}$$

By using the asymptotic formulae for the modified Bessel functions for large values of the index, the conditions for the GAPF can be written in terms of the function $F(z)$ as follows:

$$|F(z)| < \epsilon(|z|)e^{-\pi x}\left(|z|/\eta\right)^{2|y|}, \quad z = x+iy, \quad x>0, \quad |z|\to\infty, \tag{5.5}$$

where $|z|\epsilon(|z|)\to 0$ when $|z|\to\infty$. Let us denote by $z_{F,k}$, $k=1,2,\ldots$, the poles of the function $F(z)$. By making use of the fact that the zeros ω_k are simple, one finds

$$R[f(z),g(z)] = 2\pi i\left\{\sum_k \frac{\bar{I}_{i\omega_k}(\eta)F(\omega_k)}{\partial_z\bar{K}_{iz}(\eta)|_{z=\omega_k}} + r_K[f(z)]\right\}, \tag{5.6}$$

with the notation

$$\begin{aligned} r_K[f(z)] &= \sum_{k,\mathrm{Im}\,z_{F,k}\neq 0}\underset{z=z_{F,k}}{\mathrm{Res}}\frac{\bar{I}_{-iz\sigma(z_{F,k})}(\eta)}{\bar{K}_{iz}(\eta)}F(z) \\ &\quad+\sum_{k,\mathrm{Im}\,z_{F,k}=0}\underset{z=z_{F,k}}{\mathrm{Res}}\frac{\bar{I}_{iz}(\eta)+\bar{I}_{-iz}(\eta)}{2\bar{K}_{iz}(\eta)}F(z), \end{aligned} \tag{5.7}$$

and the function $\sigma(z)$ is defined in (2.2). Substituting (5.6) into (2.11), for the function $F(z)$ meromorphic in the half-plane $\mathrm{Re}\,z\geqslant a$ and satisfying the condition (5.5) one obtains the following formula

$$\begin{aligned} &\lim_{b\to+\infty}\left\{\sum_{k=m}^{n}\frac{\bar{I}_{i\omega_k}(\eta)F(\omega_k)}{\partial_z\bar{K}_{iz}(\eta)|_{z=\omega_k}} - \frac{1}{\pi^2}\mathrm{p.v.}\int_a^b dxF(x)\sinh\pi x + r_K[f(z)]\right\} \\ &\quad= -\frac{1}{2\pi i}\left[\int_a^{a+i\infty}dzF(z)\frac{\bar{I}_{-iz}(\eta)}{\bar{K}_{iz}(\eta)} - \int_a^{a-i\infty}dzF(z)\frac{\bar{I}_{iz}(\eta)}{\bar{K}_{iz}(\eta)}\right], \end{aligned} \tag{5.8}$$

where $\mathrm{Re}\,\omega_{m-1} < a < \mathrm{Re}\,\omega_m$, $\mathrm{Re}\,\omega_n < b < \mathrm{Re}\,\omega_{n+1}$, and in the expression for $r_K[f(z)]$ the summation goes over the poles with $a < \mathrm{Re}\,z_{F,k} < b$. Below we consider this formula in the limit $a\to 0$, assuming that the function $F(z)$ is meromorphic in the right half-plane $\mathrm{Re}\,z>0$.

From the relation $\bar{K}_{-iz}(\eta)=\bar{K}_{iz}(\eta)$ it follows that the purely imaginary zeros of the function $\bar{K}_{iz}(\eta)$ are complex conjugate. We will denote them by $\pm i\omega_k'$, $\omega_k'>0$, and will assume that possible purely imaginary poles of the function $F(z)$ are also complex conjugate. Let us denote the latter by $\pm iy_k$, $y_k>0$ and assume that near these poles $F(z)$ satisfies the condition

$$F(ze^{i\pi}) = -F(z) + o((z-iy_k)^{-1}), \quad z\to iy_k. \tag{5.9}$$

In the limit $a\to 0$ the rhs of formula (5.8) is presented as the sum of integrals along the straight segments of the imaginary axis between the purely imaginary poles plus the terms

$$\sum_{\alpha=+,-}\alpha\left(\int_{\gamma_\rho^\alpha} + \sum_{\sigma_k=\alpha iy_k,\alpha\omega_k'}\int_{C_\rho(\sigma_k)}\right)F(z)\frac{\bar{I}_{-\alpha iz}(\eta)}{\bar{K}_{iz}(\eta)}dz, \tag{5.10}$$

where the contours $\gamma_\rho^\pm$ and $C_\rho(\sigma_k)$ are the same as in (3.28). Assuming the relation

$$F(ze^{i\pi}) = -F(z) + o(1), \quad z\to i\omega_k', \tag{5.11}$$

in the limit $\rho\to 0$ the integrals in (5.10) are expressed in terms of the residues at the corresponding poles. Noting also that in this limit the sum of integrals over the straight segments of

the imaginary axis gives the principal value of the integral, one obtains [48]

$$\lim_{b\to+\infty}\left\{\sum_{k=1}^{n}\frac{\bar{I}_{iz}(\eta)F(z)}{\partial_z\bar{K}_{iz}(\eta)}|_{z=\omega_k,i\omega_k'}-\frac{1}{\pi^2}\text{p.v.}\int_0^b dxF(x)\sinh\pi x+r_K[f(z)]\right\}$$
$$=-\frac{1}{2\pi}\text{p.v.}\int_0^\infty dx\,\frac{\bar{I}_x(\eta)}{\bar{K}_x(\eta)}\left[F(xe^{\frac{\pi i}{2}})+F(xe^{-\frac{\pi i}{2}})\right]$$
$$-\frac{F_0\bar{I}_0(\eta)}{2\bar{K}_0(\eta)}-\sum_k\operatorname*{Res}_{z=iy_k}F(z)\frac{\bar{I}_{-iz}(\eta)}{\bar{K}_{iz}(\eta)},\tag{5.12}$$

where $F_0=\lim_{z\to0}zF(z)$. The application of this formula to physical problems will be given below in Section 18. Taking in formula (5.12) $F(z)=\bar{K}_{iz}(\eta)G(z)$, we obtain a formula relating two types of integrals with the integration over the index of the modified Bessel functions. In particular, from the latter formula it follows that for an integer m one has

$$\int_0^\infty dx\,x^{2m}\sinh\pi x\bar{K}_{ix}(\eta)\frac{\cosh ax}{\sinh bx}=\frac{\pi^2\delta_{m0}}{2b}\bar{I}_0(\eta)+\frac{\pi^2}{b}\sum_{k=1}^{\infty}(-1)^{k+m}\left(\frac{\pi k}{b}\right)^{2m}\times\cos\left(\frac{\pi ka}{b}\right)\bar{I}_{\pi k/b}(\eta),\tag{5.13}$$

under the condition $|\mathrm{Re}\,b|-|\mathrm{Re}\,a|>\pi/2$. The special case of this formula with $B=0$ and $m=0$ is given in [66].

5.2 Summation formula over the zeros of the function $Z_{iz}(u,v)$

In this subsection we will derive a summation formula over the zeros $z=\Omega_n$ of the function

$$Z_{iz}(u,v)=\bar{K}_{iz}^{(a)}(u)\bar{I}_{iz}^{(b)}(v)-\bar{I}_{iz}^{(a)}(u)\bar{K}_{iz}^{(b)}(v),\tag{5.14}$$

where we use the notation defined by formula (4.2). As we will see in the applications (see Section 19), the vacuum expectation values of physical observables in the region between two plates uniformly accelerated through the Fulling-Rindler vacuum are expressed in terms of series over these zeros. To derive a summation formula for this type of series, we choose in the GAPF the functions

$$\begin{aligned}f(z)=&\frac{2i}{\pi}\sinh\pi z\,F(z),\\ g(z)=&\frac{\bar{I}_{iz}^{(b)}(v)\bar{I}_{-iz}^{(a)}(u)+\bar{I}_{iz}^{(a)}(u)\bar{I}_{-iz}^{(b)}(v)}{Z_{iz}(u,v)}F(z),\end{aligned}\tag{5.15}$$

with a meromorphic function $F(z)$ having poles $z=z_k$ ($\neq\Omega_n$), $\mathrm{Im}\,z_k\neq0$, in the right half-plane $\mathrm{Re}\,z>0$. The zeros Ω_n are simple poles of the function $g(z)$. By taking into account the relation

$$g(z)\pm f(z)=\frac{2\bar{I}_{\mp iz}^{(a)}(u)\bar{I}_{\pm iz}^{(b)}(v)}{Z_{iz}(u,v)}F(z),\tag{5.16}$$

for the function $R[f(z),g(z)]$ in the GAPF one obtains

$$R[f(z),g(z)]=2\pi i\left[\sum_{n=1}^{\infty}\frac{\bar{I}_{-i\Omega_n}^{(b)}(v)\bar{I}_{i\Omega_n}^{(a)}(u)}{\partial_z Z_{iz}(u,v)|_{z=\Omega_n}}F(\Omega_n)+\sum_k\operatorname*{Res}_{z=z_k}\frac{F(z)}{Z_{iz}(u,v)}\bar{I}_{i\sigma(z_k)z}^{(b)}(v)\bar{I}_{-i\sigma(z_k)z}^{(a)}(u)\right],\tag{5.17}$$

where the zeros Ω_n are arranged in ascending order, and the function $\sigma(z)$ is defined in formula (2.2). Substituting relations (5.16) and (5.17) into the GAPF, as a special case the following summation formula is obtained [49, 50]:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\bar{I}_{i\Omega_n}^{(a)}(u)\bar{I}_{-i\Omega_n}^{(b)}(v)}{\partial_z Z_{iz}(u,v)|_{z=\Omega_n}} F(\Omega_n) &= \frac{1}{\pi^2}\int_0^{\infty} dx \sinh \pi x\, F(x) \\
&- \int_0^{\infty} dx \frac{F(xe^{\pi i/2}) + F(xe^{-\pi i/2})}{2\pi Z_x(u,v)} \bar{I}_x^{(a)}(u)\bar{I}_{-x}^{(b)}(v) \\
&- \sum_k \operatorname*{Res}_{z=z_k} \frac{F(z)}{Z_{iz}(u,v)} \bar{I}_{i\sigma(z_k)z}^{(b)}(v)\bar{I}_{-i\sigma(z_k)z}^{(a)}(u). \qquad (5.18)
\end{aligned}$$

Here the condition for the function $F(z)$ is obtained from the corresponding condition in the GAPF by using the asymptotic formulae for the modified Bessel function and has the form

$$|F(z)| < \epsilon(|z|)e^{-\pi x}\,(v/u)^{2|y|}\,, \quad x > 0,\ z = x + iy, \qquad (5.19)$$

for $|z| \to \infty$, where $|z|\epsilon(|z|) \to 0$ when $|z| \to \infty$. Formula (5.18) can be generalized for the case when the function $F(z)$ has real poles, under the assumption that the first integral on the right of this formula converges in the sense of the principal value. In this case the first integral on the right of Eq. (5.18) is understood in the sense of the principal value and residue terms from real poles in the form $\sum_k \operatorname*{Res}_{z=z_k} g(z)$, $\operatorname{Im} z_k = 0$, have to be added to the right-hand side of this formula, with $g(z)$ from (5.15).

6 Applications to integrals involving Bessel functions

The applications of the GAPF to infinite integrals involving combinations of Bessel functions lead to interesting results [27, 28]. First of all one can express integrals over Bessel functions through the integrals involving modified Bessel functions. Let us substitute in formula (3.18)

$$f(z) = F(z)\bar{J}_\nu(z). \qquad (6.1)$$

For the function $F(z)$ having no poles at $z = \lambda_{\nu,k}$ the sum over zeros of $\bar{J}_\nu(z)$ is zero. The conditions (3.4) and (3.11) may be written in terms of $F(z)$ as

$$|F(z)| < \varepsilon_1(x)e^{c_1|y|} \quad \text{or} \quad |F(z)| < M|z|^{-\alpha_1}e^{|y|}, \quad |z| \to \infty, \qquad (6.2)$$

with $c_1 < 1$, $x^{1/2-\delta_{B0}}\varepsilon_1(x) \to 0$ for $x \to \infty$, $\alpha_1 > \alpha_0 = 3/2 - \delta_{B0}$, and

$$F(ze^{\pi i}) = -e^{\nu\pi i}F(z) + o(z^{|\mathrm{Re}\nu|-1}). \qquad (6.3)$$

Hence, for the function $F(z)$ satisfying conditions (6.2) and (6.3), from (3.18) it follows that

$$\begin{aligned}
\text{p.v.}\int_0^{\infty} dx\, F(x)\bar{J}_\nu(x) &= r_{1\nu}\left[F(z)\bar{J}_\nu(z)\right] + \frac{\pi}{2}\operatorname*{Res}_{z=0} F(z)\bar{Y}_\nu(z) \\
&+ \frac{1}{\pi}\int_0^{\infty} dx\, \bar{K}_\nu(x)\left[e^{-\nu\pi i/2}F(xe^{\pi i/2}) + e^{\nu\pi i/2}F(xe^{-\pi i/2})\right]. \qquad (6.4)
\end{aligned}$$

In expression (3.9) for $r_{1\nu}[f(z)]$, the points z_k are poles of the meromorphic function $F(z)$, $\operatorname{Re} z_k > 0$. On the basis of Remark after Theorem 3, formula (6.4) may be generalized for the functions $F(z)$ with purely imaginary poles $\pm iy_k$, $y_k > 0$ and satisfying the condition

$$F(ze^{\pi i}) = -e^{\nu\pi i}F(z) + o\left((z \mp iy_k)^{-1}\right), \quad z \to \pm iy_k. \qquad (6.5)$$

The corresponding formula is obtained from (6.4) by adding residue terms for $z = iy_k$ in the form of (6.7) (see below) and taking the principal value of the integral on the right.

The same substitution of (6.1) into formula (3.33), with the function $F(z)$ satisfying conditions (6.2) and

$$F(xe^{\pi i/2}) = -e^{\nu\pi i}F(xe^{-\pi i/2}), \tag{6.6}$$

for real x, yields the following result

$$\text{p.v.}\int_0^\infty dx\, F(x)\bar{J}_\nu(x) = r_{1\nu}\left[F(z)\bar{J}_\nu(z)\right] + \frac{\pi i}{2}\sum_{\eta_k=0,iy_k}(2-\delta_{0\eta_k})\operatorname*{Res}_{z=\eta_k}F(z)\bar{H}_\nu^{(1)}(z). \tag{6.7}$$

In (3.9), the summation goes over the poles z_k, $\mathrm{Re}z_k > 0$ of the meromorphic function $F(z)$, and $\pm iy_k$, $y_k > 0$ are purely imaginary poles of this function. Recall that the possible real poles of $F(z)$ are such, that integral on the left of (6.7) exists.

For the functions $F(z) = z^{\nu+1}\tilde{F}(z)$, with $\tilde{F}(z)$ being analytic in the right half-plane and even along the imaginary axis, $\tilde{F}(ix) = \tilde{F}(-ix)$, one obtains

$$\int_0^\infty dx\, x^{\nu+1}\tilde{F}(x)\bar{J}_\nu(x) = 0. \tag{6.8}$$

This result for $B = 0$ (see (3.1)) has been given previously in [68]. Another result of [68] is obtained from (6.7) choosing $F(z) = z^{\nu+1}\tilde{F}(z)/(z^2 - a^2)$.

Formulae similar to (6.4) and (6.7) can be derived for the Neumann function $Y_\nu(z)$. Let for the function $F(z)$ the integral p.v.$\int_0^\infty dx\, F(x)\bar{Y}_\nu(x)$ exists. We substitute in formula (2.11)

$$f(z) = Y_\nu(z)F(z), \quad g(z) = -iJ_\nu(z)F(z), \tag{6.9}$$

and consider the limit $a \to +0$. The terms containing residues may be presented in the form

$$\begin{aligned} R[f(z), g(z)] &= \pi\sum_k \operatorname*{Res}_{\mathrm{Im}z_k>0} H_\nu^{(1)}(z)F(z) + \pi\sum_k \operatorname*{Res}_{\mathrm{Im}z_k<0} H_\nu^{(2)}(z)F(z) \\ &\quad + \pi\sum_k \operatorname*{Res}_{\mathrm{Im}z_k=0} J_\nu(z)F(z) \equiv r_{3\nu}[F(z)], \end{aligned} \tag{6.10}$$

where z_k ($\mathrm{Re}z_k > 0$) are the poles of $F(z)$ in the right half-plane. Now the following results can be proved by using (2.11):

1) If the meromorphic function $F(z)$ has no poles on the imaginary axis and satisfies condition (6.2), then

$$\begin{aligned} \text{p.v.}\int_0^\infty dx\, F(x)Y_\nu(x) &= r_{3\nu}[F(z)] - \frac{i}{\pi}\int_0^\infty dx\, K_\nu(x) \\ &\quad \times\left[e^{-\nu\pi i/2}F(xe^{\pi i/2}) - e^{\nu\pi i/2}F(xe^{-\pi i/2})\right], \end{aligned} \tag{6.11}$$

and

2) If the meromorphic function $F(z)$ satisfies the conditions

$$F(xe^{\pi i/2}) = e^{\nu\pi i}F(xe^{-\pi i/2}) \tag{6.12}$$

and (6.2), then one has

$$\text{p.v.}\int_0^\infty dx\, F(x)Y_\nu(x) = r_{3\nu}[F(z)] + \pi\sum_k \operatorname*{Res}_{z=iy_k} H_\nu^{(1)}(z)F(z), \tag{6.13}$$

where $\pm iy_k$, $y_k > 0$ are purely imaginary poles of $F(z)$.

From (6.13) it directly follows that for $F(z) = z^{\nu}\tilde{F}(z)$, with $\tilde{F}(z)$ being even along the imaginary axis, $\tilde{F}(ix) = \tilde{F}(-ix)$, and analytic in the right half-plane one has [68]:

$$\int_0^{\infty} dx\, x^{\nu}\tilde{F}(x)Y_{\nu}(x) = 0, \tag{6.14}$$

if condition (6.2) takes place.

Let us consider more general case. Let the function $F(z)$ satisfy the condition

$$F(ze^{-\pi i}) = -e^{-\lambda\pi i}F(z), \tag{6.15}$$

for $\arg z = \pi/2$. In the GAPF as functions $f(z)$ and $g(z)$ we choose

$$\begin{aligned} f(z) &= F(z)\left[J_{\nu}(z)\cos\delta + Y_{\nu}(z)\sin\delta\right], \\ g(z) &= -iF(z)[J_{\nu}(z)\sin\delta - Y_{\nu}(z)\cos\delta], \quad \delta = (\lambda-\nu)\pi/2, \end{aligned} \tag{6.16}$$

with $g(z)-(-1)^k f(z) = H_{\nu}^{(k)}(z)F(z)\exp[(-1)^k i\delta]$, $k = 1,2$. It can be seen that for such a choice the integral on the rhs of (2.11) for $a \to 0$ is equal to

$$\pi i \sum_{\eta_k = iy_k} \operatorname*{Res}_{z=\eta_k} H_{\nu}^{(1)}(z)F(z)e^{-i\delta}, \tag{6.17}$$

where $\pm iy_k$, $y_k > 0$, as above, are purely imaginary poles of $F(z)$. Substituting (6.16) into (2.11) and using (2.3) we obtain [27, 28]:

Corollary 4. *Let $F(z)$ be a meromorphic function for $\operatorname{Re} z \geqslant 0$ (except possibly at $z = 0$) with poles z_k, $\pm iy_k$; $y_k, \operatorname{Re} z_k > 0$. If this function satisfies conditions (6.2) (for $B = 0$) and (6.15) then*

$$\begin{aligned} \text{p.v.}\int_0^{\infty} dx\, F(x)\left[J_{\nu}(x)\cos\delta + Y_{\nu}(x)\sin\delta\right] = \pi i \Bigg\{ \sum_{z_k} \operatorname*{Res}_{\operatorname{Im} z_k > 0} H_{\nu}^{(1)}(z)F(z)e^{-i\delta} \\ - \sum_{z_k} \operatorname*{Res}_{\operatorname{Im} z_k < 0} H_{\nu}^{(2)}(z)F(z)e^{i\delta} - i\sum_{z_k} \operatorname*{Res}_{\operatorname{Im} z_k = 0}\left[J_{\nu}(z)\sin\delta - Y_{\nu}(z)\cos\delta\right]F(z) \\ + \sum_{\eta_k = iy_k} \operatorname*{Res}_{z=\eta_k} H_{\nu}^{(1)}(z)F(z)e^{-i\delta} \Bigg\}, \end{aligned} \tag{6.18}$$

where it is assumed that the integral on the left exists.

In particular for $\delta = \pi n$, $n = 0, 1, 2...$, formula (6.7) follows from here in the case $B = 0$. One will find many particular cases of formulae (6.7) and (6.18) looking at the standard books and tables of known integrals with Bessel functions (see, e.g., [13, 64, 67, 66, 69, 70, 71, 72]). Some special examples are given in the next section.

7 Integrals involving Bessel functions: Illustrations of general formulae

In order to illustrate the applications of general formulae from the previous section, first we consider integrals involving the function $\bar{J}_{\nu}(z)$. Let us introduce the functional

$$A_{\nu m}[G(z)] \equiv \text{p.v.}\int_0^{\infty} dz\, z^{\nu-2m-1}G(z)\bar{J}_{\nu}(z), \tag{7.1}$$

where m is an integer. Let $F_1(z)$ be an analytic function in the right half-plane satisfying the condition

$$F_1(xe^{\pi i/2}) = F_1(xe^{-\pi i/2}), \quad F_1(0) \neq 0, \tag{7.2}$$

(the case when $F_1(z) \sim z^q$, $z \to 0$, with an integer q, can be reduced to this one by redefinitions of $F_1(z)$ and m). From (6.7) the following results can be obtained [27, 28]:

$$A_{\nu m}[F_1(z)] = A^{(0)}_{\nu m}[F_1(z)] \equiv -\frac{\pi(1+\operatorname{sgn} m)}{4(2m)!}\partial_z^{2m}\left[z^{\nu}\bar{Y}_{\nu}(z)F_1(z)\right]_{z=0}, \tag{7.3}$$

$$A_{\nu m}\left[\frac{F_1(z)}{z^2-a^2}\right] = -\frac{\pi}{2}a^{\nu-2m-2}\bar{Y}_{\nu}(a)F_1(a) + A^{(0)}_{\nu m}\left[\frac{F_1(z)}{z^2-a^2}\right], \tag{7.4}$$

$$\begin{aligned} A_{\nu m}\left[\frac{F_1(z)}{z^4-a^4}\right] &= -\frac{a^{\nu-2m-4}}{2}\left[\frac{\pi}{2}\bar{Y}_{\nu}(a)F_1(a) - (-1)^m\bar{K}_{\nu}(a)F_1(ia)\right] + \\ &\quad + A^{(0)}_{\nu m}\left[\frac{F_1(z)}{z^4-a^4}\right], \end{aligned} \tag{7.5}$$

$$\begin{aligned} A_{\nu m}\left[\frac{F_1(z)}{(z^2-c^2)^{p+1}}\right] &= \frac{\pi i}{2^{p+1}p!}\left(\frac{d}{cdc}\right)^p\left[c^{\nu-2m-2}F_1(c)H^{(1)}_{\nu}(c)\right] \\ &\quad + A^{(0)}_{\nu m}\left[\frac{F_1(z)}{(z^2-c^2)^{p+1}}\right] \end{aligned} \tag{7.6}$$

$$\begin{aligned} A_{\nu m}\left[\frac{F_1(z)}{(z^2+a^2)^{p+1}}\right] &= \frac{(-1)^{m+p+1}}{2^p \cdot p!}\left(\frac{d}{ada}\right)^p\left[a^{\nu-2m-2}K_{\nu}(a)F_1(ae^{\pi i/2})\right] \\ &\quad + A^{(0)}_{\nu m}\left[\frac{F_1(z)}{(z^2+a^2)^{p+1}}\right], \end{aligned} \tag{7.7}$$

and etc. Note that $A^{(0)}_{\nu m} = 0$ for $m < 0$. In these formulae $a > 0$, $0 < \arg c < \pi/2$, and we have assumed that $\operatorname{Re}\nu > 0$. To secure convergence at the origin the condition $\operatorname{Re}\nu > m$ should be satisfied. In the last two formulae we have used the identity

$$\left(\frac{d}{dz}\right)^p\left[\frac{zF(z)}{(z+b)^{p+1}}\right]_{z=b} = \frac{1}{2^{p+1}}\left(\frac{d}{bdb}\right)^p F(b). \tag{7.8}$$

Note that (7.7) can also be obtained from (7.6) in the limit $\operatorname{Re} c \to 0$. For the case $F_1 = 1$, $m = -1$ in (7.7) see, for example, [64]. In (7.3)-(7.7) as a function $F_1(z)$ we can choose:

- function (3.51) for $\rho = 1$, $\operatorname{Re}\nu < \sum \operatorname{Re}\mu_l + 2m + (n+1)/2 - \delta_{b1} + \delta_{B0}$, $b = \sum b_l \leqslant 1$, $b_l > 0$;
- function (3.52) with $\rho = 1$, $\operatorname{Re}\nu < 2(m+n) + 1/2 - \delta_{b1} + \delta_{B0}$, $b = \sum b_l \leqslant 1$;
- function (3.53) for $\rho = 1$, $\operatorname{Re}\nu < 2m - \sum \operatorname{Re}\mu_l + (n+1)/2 - \delta_{b1} + \delta_{B0}$, $\mu_l > 0$ is half of an odd integer, $b = \sum b_l \leqslant 1$;
- function (3.54) for $\operatorname{Re}\nu < 2m + n - \sum |k_l| + 1/2 + \delta_{B0} - \delta_{\tilde{a}1}$, $\tilde{a} = \sum |a_l - b_l| \leqslant 1$, $a_l \geqslant 0$, k_l - integer.

Here we have written the conditions for (7.3). The corresponding ones for (7.4), (7.5), (7.6), (7.7) are obtained by adding on the rhs of inequalities for $\operatorname{Re}\nu$, respectively 2, 4, $2(p+1)$, $2(p+1)$. In (7.3)-(7.7) we can also choose any combinations of functions (3.51)-(3.54) with appropriate conditions.

For the evaluation of $A^{(0)}_{\nu m}$ in special cases the following formula is useful

$$\lim_{z\to 0}\left(\frac{d}{dz}\right)^{2m} f_1(z) = (2m-1)!!\lim_{z\to 0}\left(\frac{d}{zdz}\right)^m f_1(z), \tag{7.9}$$

valid for the function $f_1(z)$ satisfying condition $f_1(-z) = f_1(z) + o(z^{2m})$, $z \to 0$. From here, for instance, it follows that for $z \to 0$

$$\left(\frac{d}{dz}\right)^{2m} [z^{\nu} Y_{\nu}(bz) F_1(z)] = -(2m-1)!! \frac{2^{\nu-m}}{\pi b^{\nu-m}} \sum_{k=0}^{m} 2^k \binom{m}{k} \times \frac{\Gamma(\nu - m + k)}{b^{2k}} \left(\frac{d}{zdz}\right)^k F_1(z), \tag{7.10}$$

where we have used the standard formula for the derivative $(d/zdz)^n$ of cylinder functions (see [65]). From (7.3) one obtains ($B = 0$)

$$\int_0^{\infty} dz\, z^{\nu-2m-1} J_{\nu}(z) \prod_{l=1}^{n} \left(z^2 + z_l^2\right)^{-\mu_l/2} J_{\mu_l}(b_l \sqrt{z^2 + z_l^2}) = \frac{-\pi}{2^{m+1} m!} \times \left(\frac{d}{zdz}\right)^m \left[z^{\nu} Y_{\nu}(z) \prod_{l=1}^{n} \left(z^2 + z_l^2\right)^{-\mu_l/2} J_{\mu_l}(b_l \sqrt{z^2 + z_l^2}) \right]_{z=0}, \tag{7.11}$$

for $m \geqslant 0$ and the integral is zero for $m < 0$. Here $\operatorname{Re}\nu > 0$, $b \equiv \sum_1^n b_l \leqslant 1$, $b_l > 0$, $m < \operatorname{Re}\nu < \sum_{l=1}^n \operatorname{Re}\mu_l + 2m + (n+3)/2 - \delta_{b1}$. In the particular case $m = 0$ we obtain the Gegenbauer integral [13, 64]. In the limit $z_l \to 0$, from (7.11) the value of integral $\int_0^{\infty} dz\, z^{\nu-2m-1} J_{\nu}(z) \prod_{l=1}^n z^{-\mu_l} J_{\mu_l}(z)$ is obtained.

By using (6.18), formulae similar to (7.3)-(7.7) may be derived for the integrals of type

$$B_{\nu}[G(z)] \equiv \text{p.v.} \int_0^{\infty} dz\, G(z) \left[J_{\nu}(z) \cos\delta + Y_{\nu}(z) \sin\delta \right], \quad \delta = (\lambda - \nu)\pi/2. \tag{7.12}$$

It directly follows from Corollary 4 that for function $F(z)$ analytic for $\operatorname{Re} z \geqslant 0$ and satisfying conditions (6.2) and (6.15) the following formulae take place

$$B_{\nu}[F(z)] = 0 \tag{7.13}$$

$$B_{\nu}\left[\frac{F(z)}{z^2 - a^2}\right] = \pi F(a) \left[J_{\nu}(a) \sin\delta - Y_{\nu}(a) \cos\delta \right]/2, \tag{7.14}$$

$$B_{\nu}\left[\frac{F(z)}{z^4 - a^4}\right] = \frac{\pi}{4a^3} F(a) \left[J_{\nu}(a) \sin\delta - Y_{\nu}(a) \cos\delta \right] + \frac{i}{2a^3} K_{\nu}(a) F(ia) e^{-i\lambda\pi/2}, \tag{7.15}$$

$$B_{\nu}\left[\frac{F_1(z)}{(z^2 - c^2)^{p+1}}\right] = \frac{\pi i}{2^{p+1} \cdot p!} \left(\frac{d}{cdc}\right)^p \left[c^{-1} F(c) H_{\nu}^{(1)}(c) \right] e^{-i\delta} \tag{7.16}$$

$$B_{\nu}\left[\frac{F_1(z)}{(z^2 + a^2)^{p+1}}\right] = \frac{(-1)^{p+1}}{2^p \cdot p!} \left(\frac{d}{ada}\right)^p \left[a^{-1} F(ae^{\pi i/2}) K_{\nu}(a) \right] e^{-i\pi\lambda/2}, \tag{7.17}$$

where $a > 0$, $0 < \arg c \leqslant \pi/2$. To obtain the last two formulae we have used identity (7.8). Formula (7.13) generalizes the result of [68] (the cases $\lambda = \nu$ and $\lambda = \nu + 1$). Taking $F(z) = z^{\lambda-1}$ from the last formula we obtain the result given in [64]. In (7.13)-(7.17), as a function $F(z)$ one can choose (the constraints on the parameters are written for formula (7.13); the corresponding constraints for (7.14), (7.15), (7.16), (7.17) are obtained from the ones given by adding 2, 4, $2(p+1)$, $2(p+1)$ to the rhs of inequalities for $\operatorname{Re}\nu$, correspondingly):

- function (3.51) for $\rho = \lambda$, $|\operatorname{Re}\nu| < \operatorname{Re}\rho < \sum \operatorname{Re}\mu_l + (n+3)/2 - \delta_{b1}$, $b = \sum_l b_l \leqslant 1$;

- function (3.52) for $\rho = \lambda$, $|\mathrm{Re}\,\nu| < \mathrm{Re}\,\rho < 3/2 - \delta_{b1}$, $b = \sum_l b_l \leqslant 1$;

- function

$$z^{\rho-1}\prod_{l=1}^{n}\left[J_{\mu_l+k_l}(a_l z)Y_{\mu_l}(b_l z) - Y_{\mu_l+k_l}(a_l z)J_{\mu_l}(b_l z)\right], \quad \lambda = \rho + \sum_{l=1}^{n} k_l,$$
$$a_l > 0,\ |\mathrm{Re}\,\nu| + \sum |k_l| < \mathrm{Re}\,\rho < n + 3/2 - \delta_{c1},$$
$$c = \sum |a_l - b_l| \leqslant 1,\ \mathrm{Re}\,\mu_l \geqslant 0, \tag{7.18}$$

(for $\mathrm{Re}\,\mu_l < 0$ one has $k_l > |\mathrm{Re}\,\mu_l|$).

Any combination of these functions with appropriate conditions on parameters can be chosen as well.

Now consider integrals which can be expressed via series by using formulae (6.7) and (6.18). In (6.7) let us choose the function

$$F(z) = \frac{z^{\nu-2m}F_1(z)}{\sinh \pi z}, \tag{7.19}$$

where $F_1(z)$ is the same as in formulae (7.3)-(7.6). As the points $\pm i, \pm 2i, \ldots$ are simple poles for $F(z)$, from (6.7) one obtains

$$\int_0^\infty dz\, \frac{z^{\nu-2m}\bar{J}_\nu(z)}{\sinh(\pi z)}F_1(z) = A_{\nu m}^{(0)}\left[\frac{zF_1(z)}{\sinh(\pi z)}\right] + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{m+k}k^{\nu-2m}\bar{K}_\nu(k)F_1(ik), \tag{7.20}$$

where $A_{\nu m}^{(0)}[f(z)]$ is defined by (7.3) and $\mathrm{Re}\,\nu > m$. The corresponding constraints on $F_1(z)$ follow directly from (6.2). The particular case of this formula when $F_1(z) = \sinh(az)/z$ and $m = -1$ is given in [64]. As a function $F_1(z)$ here one can choose any of functions (3.51)-(3.54) with $\rho = 1$ and $\tilde{a}$, $\sum_l b_l < 1$. From (7.20) it follows that

$$\int_0^\infty dz\, \frac{z^{\nu-2m}}{\sinh(\pi z)}J_\nu(z)\prod_{l=1}^{n} z^{-\mu_l}I_{\mu_l}(b_l z) = A_{\nu m}^{(0)}\left[\frac{z}{\sinh(\pi z)}\prod_{l=1}^{n} z^{-\mu_l}I_{\mu_l}(b_l z)\right]$$
$$+\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{m+k}K_\nu(k)\prod_{l=1}^{n}k^{-\mu_l}J_{\mu_l}(b_l k),\ b_l > 0,\ \sum_{l=1}^{n} b_l < \pi,\ \mathrm{Re}\nu > m. \tag{7.21}$$

In a similar way the following formula can be derived from (6.18):

$$\int_0^\infty dz\, \frac{zF(z)}{\sinh(\pi z)}\left[J_\nu(z)\cos\delta + Y_\nu(z)\sin\delta\right] = \frac{2i}{\pi}e^{-i\lambda\pi/2}\sum_{k=1}^{\infty}(-1)^k kK_\nu(k)F(ik), \tag{7.22}$$

where λ is defined by relation (6.15). The constraints on the function $F(z)$ immediately follow from Corollary 4. Instead of this function we can choose functions (3.51), (3.52), (3.54).

As it has been mentioned above, adding the residue terms $\pi i \operatorname*{Res}_{z=iy_k} F(z)\bar{H}_\nu^{(1)}(z)$ to the rhs of (6.4) this formula may be generalized for functions having purely imaginary poles $\pm iy_k$, $y_k > 0$, provided condition (6.5) is satisfied. As an application let us choose

$$F(z) = \frac{z^\nu F_1(z)}{e^{2\pi z/b} - 1}, \quad F_1(-z) = F_1(z), \quad b > 0, \tag{7.23}$$

with an analytic function $F_1(z)$. Function (7.23) satisfies condition (6.5) and has poles $\pm ikb$, $k = 0, 1, 2 \ldots$. The additional constraint directly follows from (6.2). Then one obtains

$$\int_0^\infty dx\, \frac{x^\nu J_\nu(x)}{e^{2\pi x/b} - 1} F_1(x) = \frac{2}{\pi} \sum_{k=0}^{\infty}{}' (bk)^\nu K_\nu(bk) F_1(ibk) - \frac{1}{\pi} \int_0^\infty dx\, x^\nu K_\nu(x) F_1(ix), \tag{7.24}$$

where the prime indicates that the $m = 0$ term is to be halved. For the particular case $F_1(z) = 1$, using the relation

$$\sum_{k=0}^{\infty}{}' (bk)^\nu K_\nu(bk) = \frac{\sqrt{\pi}}{b} 2^\nu \Gamma(\nu + 1/2) \sum_{n=0}^{\infty}{}' \left[(2\pi n/b)^2 + 1\right]^{-\nu-1/2}, \tag{7.25}$$

and the known value for the integral on the right, we immediately obtain the result given in [64]. Relation (7.25) can be proved by using the formulae (for the integral representation of McDonald function see [64])

$$K_\nu(z) = \frac{2^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi} z^\nu} \int_0^\infty \frac{\cos zt\, dt}{(t^2 + 1)^{\nu + 1/2}}, \quad \sum_{k=-\infty}^{+\infty} e^{ikz} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(z - 2\pi n), \tag{7.26}$$

where $\delta(z)$ is the Dirac delta function.

8 Formulae for integrals involving $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$

In this section we will consider applications of the GAPF to integrals involving the function $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$. In formula (2.11) we substitute

$$f(z) = -\frac{1}{2i} F(z) \sum_{l=1}^{2} (-1)^l \frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}, \quad g(z) = \frac{1}{2i} F(z) \sum_{l=1}^{2} \frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)}. \tag{8.1}$$

For definiteness we consider the case $\lambda > 1$ (for $\lambda < 1$ the expression for $g(z)$ has to be chosen with opposite sign). Conditions (2.1) and (2.10) are satisfied if the function $F(z)$ is constrained by one of the following two inequalities

$$|F(z)| < \varepsilon(x) e^{c|y|}, \quad c < \lambda - 1,\ \varepsilon(x) \to 0,\ x \to +\infty, \tag{8.2}$$

or

$$|F(z)| < M|z|^{-\alpha} e^{(\lambda-1)|y|}, \quad \alpha > 1,\ |z| \to \infty,\ z = x + iy. \tag{8.3}$$

Then, from (2.11) it follows that for the function $F(z)$ meromorphic in $\mathrm{Re}\, z \geqslant a > 0$ one has

$$\begin{aligned} \text{p.v.}\int_0^\infty dx\, &\frac{J_\nu(x)Y_\mu(\lambda x) - J_\mu(\lambda x)Y_\nu(x)}{J_\nu^2(x) + Y_\nu^2(x)} F(x) = r_{1\mu\nu}[F(z)] \\ &+ \frac{1}{2i} \left[\int_a^{a+i\infty} dz\, F(z) \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} - \int_a^{a-i\infty} dz\, F(z) \frac{H_\mu^{(2)}(\lambda z)}{H_\nu^{(2)}(z)} \right], \end{aligned} \tag{8.4}$$

where we have introduced the notation

$$\begin{aligned} r_{1\mu\nu}[F(z)] \quad &= \quad \frac{\pi}{2} \sum_k \underset{\mathrm{Im} z_k = 0}{\mathrm{Res}} \left[F(z) \sum_{l=1}^{2} \frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)} \right] \\ &+ \pi \sum_k \sum_{l=1}^{2} \underset{(-1)^l \mathrm{Im} z_k < 0}{\mathrm{Res}} \left[F(z) \frac{H_\mu^{(l)}(\lambda z)}{H_\nu^{(l)}(z)} \right]. \end{aligned} \tag{8.5}$$

The most important case for the applications is the limit $a \to 0$. The following statements take place [27, 28]:

Theorem 5. *Let the function $F(z)$ be meromorphic for $\operatorname{Re} z \geqslant 0$ (except the possible branch point $z = 0$) with poles z_k, $\pm iy_k$ ($y_k, \operatorname{Re} z_k > 0$). If this function satisfies conditions (8.2) or (8.3) and*

$$F(xe^{\pi i/2}) = -e^{(\mu-\nu)\pi i} F(xe^{-\pi i/2}), \tag{8.6}$$

then, for values of ν for which the function $H_\nu^{(1)}(z)$ ($H_\nu^{(2)}(z)$) has no zeros for $0 \leqslant \arg z \leqslant \pi/2$ ($-\pi/2 \leqslant \arg z \leqslant 0$), the following formula is valid

$$\begin{aligned} \text{p.v.}\int_0^\infty dx\, \frac{J_\nu(x)Y_\mu(\lambda x) - Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x) + Y_\nu^2(x)} F(x) = r_{1\mu\nu}[F(z)] + \\ +\frac{\pi}{2} \sum_{\eta_k = 0, iy_k} (2 - \delta_{0\eta_k}) \operatorname*{Res}_{z=\eta_k} F(z) \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)}, \end{aligned} \tag{8.7}$$

where it is assumed that the integral on the left exists.

Proof. From condition (8.6) it follows that for $\arg z = \pi/2$ one has

$$\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z) = \frac{H_\mu^{(2)}(\lambda z e^{-\pi i})}{H_\nu^{(2)}(z e^{-\pi i})} F(z e^{-\pi i}), \tag{8.8}$$

and that the possible purely imaginary poles of $F(z)$ are conjugate: $\pm iy_k$, $y_k > 0$. Hence, on the rhs of (8.4), in the limit $a \to 0$ the term in the square brackets may be presented in the form (it can be seen in a way similar to that used for (3.32))

$$\sum_{\alpha=+,-} \left(\int_{\gamma_\rho^\alpha} + \sum_k \int_{C_\rho(\alpha i y_k)} \right) dz\, \frac{H_\mu^{(p_\alpha)}(\lambda z)}{H_\nu^{(p_\alpha)}(z)} F(z), \tag{8.9}$$

with the same notations as in (3.32). By using (8.8) and the condition that the integral converges at the origin, we obtain

$$\sum_{\alpha=+,-} \int_{\Omega_\rho^\alpha(\eta_k)} dz\, \frac{H_\mu^{(p_\alpha)}(\lambda z)}{H_\nu^{(p_\alpha)}(z)} F(z) = (2 - \delta_{0\eta_k})\, \pi i \operatorname*{Res}_{z=\eta_k} \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z), \tag{8.10}$$

where $\Omega_\rho^\pm(0) = \gamma_\rho^\pm$, $\Omega_\rho^\pm(iy_k) = C_\rho(\pm iy_k)$. By using this relation, from (8.4) we receive formula (8.7). ■

Note that one can also write the residue at $z = 0$ in the form

$$\operatorname*{Res}_{z=0} \frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)} F(z) = \operatorname*{Res}_{z=0} \frac{J_\nu(z)J_\mu(\lambda z) + Y_\nu(z)Y_\mu(\lambda z)}{J_\nu^2(z) + Y_\nu^2(z)} F(z). \tag{8.11}$$

Integrals of type (8.7) which we have found in literature (see, e.g., [13, 66, 70]) are special cases of this formula. For example, taking $F(z) = J_\nu(z)Y_{\nu+1}(\lambda' z) - Y_\nu(z)J_{\nu+1}(\lambda' z)$ for the integral on the left in (8.7) we obtain $-\lambda^{-\nu}\lambda'^{-\nu-1}$ for $\lambda' < \lambda$ and $\lambda^\nu \lambda'^{-\nu-1} - \lambda^{-\nu}\lambda'^{-\nu-1}$ for $\lambda' > \lambda$ (see

[13]). By taking $z^{2m+1}/(z^2+a^2)$, $z^{2m+1}/(z^2-c^2)$ as $F(z)$, for $\mu=\nu$ and integer $m\geqslant 0$ one obtains

$$\int_0^\infty dx\,\frac{J_\nu(x)Y_\nu(\lambda x)-Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)}\frac{x^{2m+1}}{x^2+a^2} = (-1)^m a^{2m}\frac{\pi}{2}\frac{K_\nu(\lambda a)}{K_\nu(a)}, \tag{8.12}$$

$$\begin{aligned}\text{p.v.}\int_0^\infty dx\,\frac{J_\nu(x)Y_\nu(\lambda x)-Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)}\frac{x^{2m+1}}{x^2-c^2} &= \frac{\pi}{2}c^{2m}\\ &\times\frac{J_\nu(c)J_\nu(\lambda c)+Y_\nu(c)Y_\nu(\lambda c)}{J_\nu^2(c)+Y_\nu^2(c)},\end{aligned} \tag{8.13}$$

where $\mathrm{Re}\,a>0$, $c>0,\lambda>1$. Special cases of this formula for $\nu=m=0$ are given in [13]. In (8.12) taking the limit $a\to 0$ and choosing $m=0$, we obtain the integral of this type given in [66]. In (8.7) as a function $F(z)$ we can choose (3.51), (3.52), (3.54) (the corresponding conditions for parameters directly follow from (8.2) or (8.3)) with $\rho=\mu-\nu-2m$ (m is an integer), as well as any products between them and with $\prod_{l=1}^n(z^2-c_l^2)^{-k_l}$. For instance,

$$\begin{aligned}&\int_0^\infty \frac{dx}{x}\frac{J_\nu(x)Y_\nu(\lambda x)-Y_\nu(x)J_\nu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)}\prod_{l=1}^n\frac{J_{\mu_l}(b_l\sqrt{x^2+z_l^2})}{(x^2+z_l^2)^{\mu_l/2}}=\frac{\pi}{2\lambda^\nu}\\ &\times\prod_{l=1}^n z^{-\mu_l}J_{\mu_l}(b_lz_l),\ b_l,\ \mathrm{Re}\,\nu>0,\ \mathrm{Re}\,z_l\geqslant 0,\ \lambda>1,\\ &\sum_{l=1}^n\mathrm{Re}\,\mu_l+n/2+1>\delta_{b,\lambda-1},\ b\equiv\sum_{l=1}^n b_l\leqslant\lambda-1.\end{aligned} \tag{8.14}$$

As another consequence of formula (8.4) one has:

Theorem 6. *Let $F(z)$ be meromorphic in the right half-plane (with possible exception $z=0$) with poles z_k, $\mathrm{Re}\,z_k>0$, and satisfy conditions (8.2) or (8.3) and*

$$F(ze^{\pi i})=-e^{(\mu-\nu)\pi i}F(z)+o(z^{|\mathrm{Re}\mu|-|\mathrm{Re}\nu|-1}),\quad z\to 0, \tag{8.15}$$

then for values of ν for which the function $H_\nu^{(1)}(z)$ ($H_\nu^{(2)}(z)$) has no zeros for $0\leqslant\arg z\leqslant\pi/2$ ($-\pi/2\leqslant\arg z\leqslant 0$) the following formula takes place

$$\begin{aligned}&\text{p.v.}\int_0^\infty dx\,\frac{J_\nu(x)Y_\mu(\lambda x)-Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)}F(x)=r_{1\mu\nu}[F(z)]+\frac{\pi}{2}\operatorname*{Res}_{z=0}\frac{H_\mu^{(1)}(\lambda z)}{H_\nu^{(1)}(z)}F(z)\\ &+\frac{1}{2}\int_0^\infty dx\,\frac{K_\mu(\lambda x)}{K_\nu(x)}\left[e^{(\nu-\mu)\pi i/2}F(xe^{\pi i/2})+e^{(\mu-\nu)\pi i/2}F(xe^{-\pi i/2})\right],\ \lambda>1,\end{aligned} \tag{8.16}$$

provided the integral on the left exists.

Proof. This result immediately follows from (8.4) in the limit $a\to 0$ and from (8.10) with $\eta_k=0$. ■

For example, by using (8.16) one obtains

$$\begin{aligned}&\int_0^\infty dx\frac{J_\nu(x)Y_\mu(\lambda x)-Y_\nu(x)J_\mu(\lambda x)}{J_\nu^2(x)+Y_\nu^2(x)}\prod_{l=1}^n J_{\mu_l}(b_lx)=\cos\mu_s\int_0^\infty dx\,\frac{K_\mu(\lambda x)}{K_\nu(x)}\\ &\times\prod_{l=1}^n I_{\mu_l}(b_lx),\ \sum_{l=1}^n\mathrm{Re}\,\mu_l+|\mathrm{Re}\,\nu|>|\mathrm{Re}\,\mu|-1,\ b=\sum_{l=1}^n b_l\leqslant\lambda-1,\ b_l>0,\\ &n>\delta_{b,\lambda-1},\ \mu_s\equiv\nu-\mu+\sum_{l=1}^n\mu_l.\end{aligned} \tag{8.17}$$

Such relations are useful in numerical calculations of the integrals on the left as the integrand on the right at infinity goes to zero exponentially fast.

We have considered formulae for integrals containing $J_\nu(z)Y_\mu(\lambda z) - Y_\nu(z)J_\mu(\lambda z)$. Similar results can be obtained for integrals with the functions $J'_\nu(z)Y_\mu(\lambda z) - Y'_\nu(z)J_\mu(\lambda z)$ and $J'_\nu(z)Y'_\mu(\lambda z) - Y'_\nu(z)J'_\mu(\lambda z)$.

9 Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor on manifolds with boundaries

In this and following sections we will consider applications of the GAPF to the problems in quantum field theory with boundaries, mainly to the Casimir effect. The Casimir effect is one of the most interesting macroscopic manifestations of the non-trivial structure of the vacuum state in quantum field theory [2, 3, 4, 5, 6, 7]. The effect is a phenomenon common to all systems characterized by fluctuating quantities and results from changes in the vacuum fluctuations of a quantum field that occur because of the imposition of boundary conditions or the choice of topology. It may have important implications on all scales, from cosmological to subnuclear, and has become in recent decades an increasingly popular topic in quantum field theory. In particular, the recent measurements of the Casimir forces between macroscopic bodies (see, for instance, [6, 73]) provide a sensitive test for constraining the parameters of long-range interactions predicted by modern unification theories of fundamental interactions [74]. In addition to its fundamental interest the Casimir effect also plays an important role in the fabrication and operation of nano- and micro-scale mechanical systems [6].

Before to consider special problems we first give general introduction for the procedure of the evaluation of the vacuum expectation values (VEVs) of the functions bilinear in the field on manifolds with boundaries. Consider a real scalar field φ with curvature coupling parameter ζ on a $(D+1)$-dimensional background spacetime $\mathcal{M}$ with boundary $\partial\mathcal{M}$ described by the metric tensor g_{ik}. The corresponding field equation in the bulk has the form

$$\left(\nabla_i\nabla^i + m^2 + \zeta R\right)\varphi = 0, \tag{9.1}$$

where R is the scalar curvature for the background spacetime, m is the mass for the field quanta, ∇_i the covariant derivative operator associated with the metric g_{ik}. On the boundary of the manifold the field satisfies some prescribed boundary condition $F[\varphi(x)]|_{x\in\partial\mathcal{M}} = 0$. Here and below we adopt the conventions of Birrell and Davies [75] for the metric signature and the curvature tensor. The values of the curvature coupling parameter $\zeta = 0$ and $\zeta = \zeta_D$ with $\zeta_D \equiv (D-1)/4D$ correspond to the most important special cases of the minimal and conformal couplings. It is convenient for later use to write the corresponding metric energy-momentum tensor (EMT) in the form

$$T_{ik} = \nabla_i\varphi\nabla_k\varphi + \left[\left(\zeta - \frac{1}{4}\right)g_{ik}\nabla_l\nabla^l - \zeta\nabla_i\nabla_k - \zeta R_{ik}\right]\varphi^2, \tag{9.2}$$

with R_{ik} being the Ricci tensor. This form of the EMT differs from the standard one by the term $\varphi g_{ik}\left(\nabla_l\nabla^l + m^2 + \zeta R\right)\varphi/2$ which vanishes on solutions of the field equation (see, for instance, [76]) . It can be easily checked that the following relation for the trace of the EMT takes place:

$$T_i^i = D(\zeta - \zeta_D)\nabla_i\nabla^i\varphi^2 + m^2\varphi^2. \tag{9.3}$$

Note that on manifolds with boundaries the EMT in addition to bulk part (9.2) contains a contribution located on the boundary. For arbitrary bulk and boundary geometries the expression

for the surface EMT is derived in [76] by using the standard variational procedure for the action with boundary term. For solutions of Eq. (9.1), φ_1 and φ_2, the scalar product is defined by the relation

$$(\varphi_1, \varphi_2) = -i \int_{\Sigma} d\Sigma \, \sqrt{-g_{\Sigma}} n^i \left[\varphi_1 \partial_i \varphi_2^* - (\partial_i \varphi_1) \, \varphi_2^* \right], \tag{9.4}$$

where $d\Sigma$ is the volume element on a given spacelike hypersurface Σ, n^i is the timelike unit vector normal to this hypersurface, and g_{Σ} is the determinant of the induced metric. The quantization of the field can be done by using the standard canonical quantization procedure (see, for instance, [2, 75, 77, 78]).

Let $\{\varphi_{\sigma}(x), \varphi_{\sigma}^*(x)\}$ is a complete orthonormal set of classical solutions to the field equation satisfying the boundary condition,

$$(\varphi_{\sigma}(x), \varphi_{\sigma'}(x)) = \delta_{\sigma\sigma'}, \ F[\varphi_{\sigma}(x)]|_{x \in \partial \mathcal{M}} = 0, \tag{9.5}$$

where the collective index σ can contain both discrete and continuous components. On the right of the orthonormalization condition, $\delta_{\sigma\sigma'}$ is understood as the Kronecker symbol for discrete components and the Dirac delta function for continuous ones. In quantum field theory we expand the field operator in terms of the complete set:

$$\varphi(x) = \sum_{\sigma} [a_{\sigma} \varphi_{\sigma}(x) + a_{\sigma}^{+} \varphi_{\sigma}^*(x)], \tag{9.6}$$

with annihilation and creation operators a_{σ}, a_{σ}^{+}, satisfying the commutation relation $\left[a_{\sigma}, a_{\sigma'}^{+}\right] = \delta_{\sigma\sigma'}$. The vacuum state $|0\rangle$ is defined by the relation $a_{\sigma}|0\rangle = 0$ for any σ. Our main interest below will be the VEVs of the field square and the EMT. These functions give comprehensive insight into vacuum fluctuations and are among the most important quantities characterizing the properties of the quantum vacuum. Though the corresponding operators are local, due to the global nature of the vacuum, the VEVs describe both local and global properties of the bulk and carry an important information about the background geometry. In addition to describing the physical structure of the quantum field at a given point, the VEV of the EMT acts as the source of gravity in the Einstein equations, and therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field [75].

The VEVs of the field square and the EMT can be obtained from a two-point function taking the coincidence limit in combination with the renormalization procedure. As a two-point function we will take the positive frequency Wightman function defined as the VEV

$$W(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle. \tag{9.7}$$

Our choice of this function is related to that it also determines the response of the Unruh-DeWitt type particle detector at a given state of motion [75]. Substituting into formula (9.7) the expansion of the field operator (9.6) and using the commutation relation for the annihilation and creation operators, the following mode-sum formula is obtained

$$W(x, x') = \sum_{\sigma} \varphi_{\sigma}(x)\varphi_{\sigma}^*(x'). \tag{9.8}$$

We can evaluate the VEVs of the field square and the energy-momentum tensor as the following coincidence limits:

$$\langle 0|\varphi^2(x)|0\rangle = \lim_{x' \to x} W(x, x'), \tag{9.9}$$

$$\begin{aligned} \langle 0|T_{ik}(x)|0\rangle &= \lim_{x' \to x} \partial_i \partial'_k W(x, x') \\ &\quad + \left[\left(\zeta - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \zeta \nabla_i \nabla_k - \zeta R_{ik} \right] \langle 0|\varphi^2(x)|0\rangle, \end{aligned} \tag{9.10}$$

where ∂_i is the derivative with respect to x^i and ∂'_k is the derivative with respect to x'^k. Similar formulae can be written for fermionic and electromagnetic fields. The expressions on the right of formulae (9.9), (9.10) are formal as they diverge and to obtain a finite physical result some renormalization procedure is needed. The consideration of the factors in the field bilinear products at different spacetime points corresponds to the point-splitting regularization procedure. An alternative way is to introduce in the mode-sums of the field square and the energy-momentum tensor a cutoff function, which makes them convergent, and to remove this function after the renormalization. Other regularization methods for the VEVs of the field square and the EMT are based on the use of the Green function (for the Green function method in the Casimir effect calculations see [7] and references therein) and the local zeta function [79] (see also [80]). The discussion for the relations between various regularization techniques can be found in [2, 3, 7, 75, 80].

The important thing is that, for points away from boundaries the divergences in the VEVs of the field square and the EMT are the same as those on background of manifolds without boundaries and, hence, for these points the corresponding renormalization procedure is also the same. However, the local VEVs diverge for the points on the boundary. These surface divergences are well known in quantum field theory with boundaries and are investigated for various types of bulk and boundary geometries (see, for example, [75, 77, 81, 82, 83, 84]). A powerful tool for studying one-loop divergences in the VEVs is the heat-kernel expansion [80, 85, 86]. In general, the physical quantities in problems with boundary conditions can be classified into two main groups. The first group includes quantities which do not contain surface divergences. For these quantities the renormalization procedure is the same as in quantum field theory without boundaries and they can be evaluated by boundary condition calculations. The contribution of the higher modes into the boundary induced effects in these quantities is suppressed by parameters already present in the idealized model. Examples of such quantities are the energy density and the vacuum stresses at the points away from the boundary and the interaction forces between disjoint bodies. For quantities from the second group, such as the energy density on the boundary and the total vacuum energy, the contribution of the arbitrary higher modes is dominant and they contain divergences which cannot be eliminated by the standard renormalization procedure of quantum field theory without boundaries. Of course, the model where the physical interaction is replaced by the imposition of boundary conditions on the field for all modes is an idealization. The appearance of divergences in the process of the evaluation of physical quantities of the second type indicate that more realistic physical model should be employed for their evaluation. In literature on the Casimir effect different field-theoretical approaches have been discussed to extract the finite parts from the diverging quantities. However, in the physical interpretation of these results it should be taken into account that these terms are only a part of the full expression of the physical quantity and the terms which are divergent in the idealized model can be physically essential and their evaluation needs a more realistic model. It seems plausible that such effects as surface roughness, or the microstructure of the boundary on small scales can introduce a physical cutoff needed to produce finite values for surface quantities.

Below we are interested in the VEVs of the field square and the EMT at points away from boundaries. These VEVs belong to the first group of quantities and, hence, they are well-defined within the framework of standard renormalization procedure of quantum field theory without boundaries. We expect that the same results will be obtained in the model where instead of externally imposed boundary condition the fluctuating field is coupled to a smooth background potential that implements the boundary condition in a certain limit [9]. For a scalar field we

will assume that on the boundary the field satisfies Robin boundary condition

$$\left(\tilde{A}+\tilde{B}n^i\nabla_i\right)\varphi(x)=0,\ x\in\partial\mathcal{M}, \tag{9.11}$$

where $\tilde{A}$ and $\tilde{B}$ are constants, and n^i is the unit inward-pointing normal to the boundary. Of course, all results will depend only on the ratio of the coefficients in (9.11). However, to keep the transition to Dirichlet and Neumann cases transparent we will write the boundary condition in the form (9.11). Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and appear in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields, spinor and gauge field theories, quantum gravity, supergravity and braneworld scenarios [56, 87, 88, 89]. In some geometries, Robin boundary conditions may be useful for depicting the finite penetration of the field into the boundary with the 'skin-depth' parameter related to the Robin coefficient [90, 91]. In examples with fermionic fields we will consider the bag boundary condition and in the case of the electromagnetic field we assume perfect conductor boundary conditions.

10 Examples for the application of the Abel-Plana formula

10.1 Casimir effect in $R^D\times S^1$

First we consider an example of the application of the standard Abel-Plana summation formula for the evaluation of the vacuum characteristics in the flat spacetime assuming that one of the spatial coordinates, the coordinate $x^D=y$, is compactified to a circle with the length a and the scalar field satisfies periodic boundary condition $\varphi(y+a)=\varphi(y)$ (for the Casimir effect in topologically non-trivial spaces and its role in cosmological models see [3, 6, 7, 92] and references therein). The corresponding normalized eigenfunctions have the form

$$\varphi_\sigma=\frac{e^{ik_ny+i\mathbf{k}\cdot\mathbf{x}-i\omega_nt}}{[2\omega_na(2\pi)^{D-1}]^{1/2}},\ k_n=\frac{2\pi}{a}n,\ n=0,\pm1,\pm2,\ldots, \tag{10.1}$$

and $\omega_n^2=k_n^2+k^2+m^2$. The vector $\mathbf{x}=(x^1,x^2,\ldots,x^{D-1})$ specifies the uncompactified spatial dimensions and $k=|\mathbf{k}|$. From mode-sum formula (9.8) for the Wightman function one finds

$$W_{\mathrm{P}}(x,x',a)=\frac{1}{a}\int d\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}}}{(2\pi)^{D-1}}{\sum_{n=0}^{\infty}}'\frac{e^{-i\omega_n\Delta t}}{\omega_n}\cos(k_n\Delta y), \tag{10.2}$$

where $\Delta\mathbf{x}=\mathbf{x}-\mathbf{x}'$, $\Delta t=t-t'$, $\Delta y=y-y'$. We apply to the series over n the APF in the form (2.16). The corresponding conditions are satisfied if $|\Delta y|+|\Delta t|<2a$. In particular, this is the case in the coincidence limit assuming that $|y|<a$. It is easily seen that the contribution coming from the first integral on the right of the APF is the Wightman function for the Minkowski spacetime, $W_{\mathrm{M}}(x,x')$. As a result, the Wightman function is written in the form

$$W_{\mathrm{P}}(x,x',a)=W_{\mathrm{M}}(x,x')+2\int d\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}}}{(2\pi)^D}\int_{k_m}^{\infty}dx\,\frac{\cosh(x\Delta y)}{e^{ax}-1}\frac{\cosh(\Delta t\sqrt{x^2-k_m^2})}{\sqrt{x^2-k_m^2}}, \tag{10.3}$$

with the notation $k_m=\sqrt{k^2+m^2}$. In the second term on the right of this formula we first integrate over the angular part of the vector $\mathbf{k}$. Further, we introduce polar coordinates in the plane (k,x) and integrate over the polar angle. The corresponding integrals can be found in

[66]. In this way we obtain the formula

$$
\begin{aligned}
W_{\mathrm{P}}(x,x',a) &= W_{\mathrm{M}}(x,x') + \frac{(2\pi)^{-\frac{D}{2}}}{(\Delta z)^{\frac{D-2}{2}}} \int_m^\infty dx \, \frac{(x^2-m^2)^{\frac{D-2}{4}}}{e^{ax}-1} \\
&\quad \times \cosh(x\Delta y) J_{D/2-1}(\Delta z \sqrt{x^2-m^2}), \qquad (10.4)
\end{aligned}
$$

where $\Delta z = [|\Delta \mathbf{x}|^2 - (\Delta t)^2]^{1/2}$. By using the expansion $(e^{ax}-1)^{-1} = \sum_{n=1}^\infty e^{-anx}$, in (10.4) the separate terms in the series are explicitly integrated and we find an equivalent representation

$$
W_{\mathrm{P}}(x,x',a) = \frac{m^{D-1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{n=-\infty}^{+\infty} \frac{K_{(D-1)/2}(u_n)}{u_n^{(D-1)/2}}, \qquad (10.5)
$$

with $u_n = m[|\Delta \mathbf{x}|^2 + (\Delta y + an)^2 - (\Delta t)^2]^{1/2}$. The $n = 0$ term in this formula is the Minkowskian Wightman function. Formula (10.5) presents the Wightman function as an image sum of the corresponding Minkowskian functions.

The renormalized VEV of the field square is obtained subtracting from the Wightman function the Minkowskian part and taking the coincidence limit. As we have already separated the latter, this procedure here is trivial and one obtains

$$
\begin{aligned}
\langle \varphi^2 \rangle_{\mathrm{ren}}^{(\mathrm{P})} &= \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)} \int_m^\infty dx \, \frac{(x^2-m^2)^{D/2-1}}{e^{ax}-1} \\
&= \frac{2m^{D-1}}{(2\pi)^{(D+1)/2}} \sum_{n=1}^\infty \frac{K_{(D-1)/2}(man)}{(man)^{(D-1)/2}}. \qquad (10.6)
\end{aligned}
$$

In the similar way, for the components of the vacuum EMT we find (no summation over i)

$$
\begin{aligned}
\langle T_i^i \rangle_{\mathrm{ren}}^{(\mathrm{P})} &= -\frac{2^{1-D}}{\pi^{D/2} D\Gamma(D/2)} \int_m^\infty dx \, \frac{(x^2-m^2)^{D/2}}{e^{ax}-1} \\
&= -\frac{2m^{D+1}}{(2\pi)^{(D+1)/2}} \sum_{n=1}^\infty \frac{K_{(D+1)/2}(man)}{(man)^{(D+1)/2}}, \qquad (10.7)
\end{aligned}
$$

for $i = 0, 1, \ldots, D-1$, and the component $\langle T_D^D \rangle_{\mathrm{ren}}^{(\mathrm{P})}$ is obtained from the trace relation (9.3): $\langle T_l^l \rangle_{\mathrm{ren}}^{(\mathrm{P})} = m^2 \langle \varphi^2 \rangle_{\mathrm{ren}}^{(\mathrm{P})}$. We have considered periodic boundary conditions. For the field with antiperiodic conditions (twisted scalar field) we have $\varphi(y+a) = -\varphi(y)$. Now, in the expressions of the eigenfunctions $k_n = 2\pi(n+1/2)/a$. The corresponding VEVs are evaluated in the way similar to that for the periodic case with the application of the APF in the form (2.19). Already on these simple examples we have seen two important features of the application of the APF. First, this enables to extract from the VEVs the part corresponding to the geometry with decompactified dimensions and second, to present the parts induced by the non-trivial topology in terms of rapidly convergent integrals.

10.2 Casimir effect for parallel plates with Dirichlet and Neumann boundary conditions

In considered examples, due to the homogeneity of the background spacetime the VEVs of the field square and the EMT do not depend on coordinates. This is not the case in the Minkowski spacetime with boundaries. The presence of boundaries breaks the homogeneity and leads to coordinate-dependent VEVs. Simplest examples of this kind are scalar fields with Dirichlet and

Neumann boundary conditions on two parallel plates with distance a. Assuming that the plates are located at $y=0$ and $y=a$, the corresponding eigenfunctions are obtained from (10.1) by the replacement $e^{ik_n y} \to \sqrt{2}\sin(k_n y)$ with $k_n = \pi n/a$, $n=1,2,\ldots$, for Dirichlet case, and by the replacement $e^{ik_n y} \to \sqrt{2-\delta_{n0}}\cos(k_n y)$ with $k_n = \pi n/a$, $n=0,1,2,\ldots$, for Neumann one. Now, applying to the series over n in the mode-sum for the Wightman function $W_{\mathrm{J}}(x,x')$ (with $\mathrm{J}=\mathrm{D},\mathrm{N}$ for Dirichlet and Neumann scalars) the APF (2.16), we can see that the term with the first integral on the right of the APF corresponds to the Wightman function $W_{\mathrm{J}}^{(0)}(x,x')$ for the geometry of a single plate at $y=0$, and the term with the second integral is the part induced by the presence of the second plate at $y=a$. It can be seen that the following relations take place

$$
\begin{aligned}
W_{\mathrm{J}}^{(0)}(x,x') &= W_{\mathrm{M}}(x,x') + \delta_{\mathrm{J}} W_{\mathrm{M}}(x,x'_{\mathrm{I}}), && (10.8)\\
W_{\mathrm{J}}(x,x') &= W_{\mathrm{P}}(x,x',2a) + \delta_{\mathrm{J}} W_{\mathrm{P}}(x,x'_{\mathrm{I}},2a), && (10.9)
\end{aligned}
$$

with $\delta_{\mathrm{D}}=-1$, $\delta_{\mathrm{N}}=1$, and x'_{I} is the image for the point x' with respect to the plate at $y=0$: $x'_{\mathrm{I}} = (t,\mathbf{x}',-y')$.

For the geometry of a single plate at $y=0$, by using the Wightman function (10.8) one finds

$$
\langle\varphi^2\rangle_{\mathrm{J,ren}}^{(0)} = \frac{\delta_{\mathrm{J}} m^{D-1}}{(2\pi)^{(D+1)/2}} \frac{K_{(D-1)/2}(2m|y|)}{(2m|y|)^{(D-1)/2}}, \tag{10.10}
$$

for the VEV of the field square, and (no summation over i)

$$
\langle T_i^i\rangle_{\mathrm{J,ren}}^{(0)} = \frac{4\delta_{\mathrm{J}} m^{D+1}(\zeta-\zeta_D)}{(2\pi)^{(D+1)/2}} \left[\frac{K_{(D+1)/2}(2m|y|)}{(2m|y|)^{(D+1)/2}} - \frac{K_{(D+3)/2}(2m|y|)}{(2m|y|)^{(D-1)/2}}\right] + \frac{m^2}{D}\langle\varphi^2\rangle_{\mathrm{J,ren}}^{(0)}, \tag{10.11}
$$

$i=0,1,\ldots,D-1$, $\langle T_D^D\rangle_{\mathrm{J,ren}}^{(0)}=0$, for the VEV of the EMT. In particular, for a conformally coupled massless scalar the latter vanishes. For points on the plate the VEVs diverge with the leading terms

$$
\langle\varphi^2\rangle_{\mathrm{J,ren}}^{(0)} \approx \frac{\delta_{\mathrm{J}}\Gamma((D-1)/2)}{(4\pi)^{(D+1)/2}|y|^{D-1}},\quad \langle T_i^k\rangle_{\mathrm{J,ren}}^{(0)} \approx -\frac{\delta_i^k D\delta_{\mathrm{J}}(\zeta-\zeta_D)}{2^D\pi^{(D+1)/2}|y|^{D+1}}\Gamma\left(\frac{D+1}{2}\right), \tag{10.12}
$$

for $i=0,1,\ldots,D-1$. Note that the expressions on the right of these formulae are the VEVs of the field square and the EMT for a massless scalar.

For the geometry of two plates at $y=0$ and $y=a$ the VEVs are presented in the form

$$
\langle\varphi^2\rangle_{\mathrm{J,ren}} = \langle\varphi^2\rangle_{\mathrm{J,ren}}^{(0)} + \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)}\int_m^\infty dx\, \frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1}\left[1+\delta_{\mathrm{J}}\cosh(2xy)\right], \tag{10.13}
$$

for the field square, and (no summation over i)

$$
\begin{aligned}
\langle T_i^i\rangle_{\mathrm{J,ren}} &= \langle T_i^i\rangle_{\mathrm{J,ren}}^{(0)} - \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)}\int_m^\infty dx\, \frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1}\\
&\quad\times\left\{\frac{x^2-m^2}{D} + \delta_{\mathrm{J}}\left[4(\zeta-\zeta_D)x^2 - m^2/D\right]\cosh(2xy)\right\}, && (10.14)\\
\langle T_D^D\rangle_{\mathrm{J,ren}} &= \frac{2^{1-D}}{\pi^{D/2}\Gamma(D/2)}\int_m^\infty dx\, x^2\frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1}, && (10.15)
\end{aligned}
$$

$i=0,1,\ldots,D-1$, for the EMT. Note that by using the formula

$$
\begin{aligned}
\int_m^\infty dx\, x^p\frac{(x^2-m^2)^{D/2-1}}{e^{2ax}-1}\cosh(2xy) &= (-1)^p\frac{m^{D+p-1}}{\sqrt{\pi}}2^{\frac{D-3}{2}}\Gamma\left(\frac{D}{2}\right)\\
&\quad\times\sum_{n=-\infty}^{\infty}{}' \partial_z^p\frac{K_{(D-1)/2}(z)}{z^{(D-1)/2}}|_{z=2m|an+y|}, && (10.16)
\end{aligned}
$$

where the prime means that the $n=0$ term should be omitted, we can present the corresponding results in terms of series containing the McDonald function. The vacuum forces acting on the plates are determined by the component $\langle T_D^D \rangle_{\mathrm{J,ren}}$ and they are the same for Dirichlet and Neumann scalars. Formula (10.15) for the vacuum forces acting on the plates can also be obtained by differentiating the corresponding vacuum energy from [93] (see also [7]). The forces are attractive for all values of the interplate distance. In particular, in the case of a massless field for the modulus of the force acting per unit surface of the plate one finds

$$F_{\mathrm{J}} = \frac{D\zeta_{\mathrm{R}}(D+1)}{(4\pi)^{(D+1)/2} a^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \ \mathrm{J} = \mathrm{D}, \mathrm{N}, \tag{10.17}$$

where $\zeta_{\mathrm{R}}(x)$ is the Riemann zeta function. Local analysis of a quantum scalar field confined within rectangular cavities is presented in [94].

In the dimension $D=3$ with perfect conductor boundary condition on two parallel plates, the electromagnetic field is presented as a set of two types of modes corresponding to massless Dirichlet and Neumann scalars. The corresponding VEV for the EMT does not depend on coordinates and in the region between the plates is given by [95]

$$\langle T_i^k \rangle_{\mathrm{El,ren}} = -\frac{\pi^2}{720 a^4} \mathrm{diag}(1,1,1,-3). \tag{10.18}$$

Due to the cancellation between Dirichlet and Neumann modes, the electromagnetic EMT is uniform in the region between the plates and vanishes in the outside regions.

11 Casimir effect for parallel plates with Robin boundary conditions

As we have demonstrated in the previous section the application of the APF provides an efficient way for the investigation of the boundary-induced effects. Here we will see that already in the case of two parallel plate geometry with Robin boundary conditions this procedure needs a generalization [32]. This can be done by using the GAPF. As before, we assume that the plates are located at $y = a_1 = 0$ and $y = a_2 = a$, and the field obeys boundary conditions (9.11) with coefficients $\tilde{A}_j$, $\tilde{B}_j$ for the plate at $y = a_j$. In the region between the plates the eigenfunctions are presented in the form

$$\varphi_\sigma = \beta(k_y) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(k_y)t} \cos\left(k_y y + \alpha(k_y)\right), \tag{11.1}$$

where $\omega(k_y) \equiv \sqrt{k_y^2 + k^2 + m^2}$, and $\alpha(k_y)$ is defined by the relation

$$e^{2i\alpha(k_y)} \equiv \frac{i\beta_1 k_y - 1}{i\beta_1 k_y + 1}, \ \beta_j = (-1)^{j-1} \tilde{B}_j / \tilde{A}_j. \tag{11.2}$$

The corresponding eigenvalues for k_y are obtained from the boundary conditions and are solutions of the following transcendental equation:

$$F(z) \equiv \left(1 - b_1 b_2 z^2\right) \sin z - (b_1 + b_2)\, z \cos z = 0, \ z = k_y a, \ b_j = \beta_j / a. \tag{11.3}$$

The expression for the coefficient $\beta(k_y)$ in (11.1) is obtained from the normalization condition:

$$\beta^{-2}(k_y) = (2\pi)^{D-1} a\omega \left[1 + \frac{\sin(k_y a)}{k_y a} \cos(k_y a + 2\alpha(k_y))\right]. \tag{11.4}$$

The eigenvalue equation (11.3) has an infinite set of real zeros which we will denote by $k_y = \lambda_n/a$, $n = 1, 2, \ldots$. In addition, depending on the values of the coefficients in the boundary conditions, this equation has two or four complex conjugate purely imaginary zeros $\pm iy_l$, $y_l > 0$ (see [32]).

Substituting eigenfunctions (11.1) into mode-sum formula (9.8), for the positive-frequency Wightman function in the region between two plates one finds

$$\begin{aligned} W(x,x') &= \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \sum_{k_y=\lambda_n/a, iy_l/a} \beta^2(k_y) e^{-i\omega(k_y)\Delta t} \\ &\quad \times \cos(k_y y + \alpha(k_y)) \cos(k_y y' + \alpha(k_y)). \end{aligned} \tag{11.5}$$

To obtain the summation formula over the zeros λ_n, in the GAPF as a function $g(z)$ we choose

$$g(z) = -i\left[\left(1 - b_1 b_2 z^2\right)\cos z + (b_1 + b_2)\, z \sin z\right] \frac{f(z)}{F(z)}. \tag{11.6}$$

Substituting (11.6) into (2.11) and taking the limit $a \to 0$, under the assumption that the poles $\pm iy_l$ are excluded by small semicircles on the right half-plane, one obtains the following summation formula [32]

$$\begin{aligned} \sum_{z=\lambda_n, iy_l} \frac{\pi f(z)}{1 + \cos(z + 2\alpha)\sin z/z} &= \frac{\pi f(0)/2}{b_1 + b_2 - 1} + \int_0^\infty dz f(z) \\ &\quad + i\int_0^\infty dt \frac{f(te^{\pi i/2}) - f(te^{-\pi i/2})}{\frac{(b_1 t - 1)(b_2 t - 1)}{(b_1 t + 1)(b_2 t + 1)} e^{2t} - 1} \\ &\quad - \frac{\theta(b_1)}{2b_1}\left[h(e^{\frac{\pi i}{2}}/b_1) + h(c_1 e^{-\frac{\pi i}{2}}/b_1)\right], \end{aligned} \tag{11.7}$$

where $h(z) \equiv \left(b_1^2 z^2 + 1\right) f(z)$. For the case $b_2 = 0$, $b_1 < 0$, with an analytic function $f(z)$ this formula is derived in [91].

For the summation over the eigenmodes λ_n in (11.5) as a function $f(z)$ in (11.7) we take

$$f(z) \equiv \frac{e^{-i\omega(z/a)\Delta t}}{a\omega(z/a)} \cos(zy/a + \alpha(z/a)) \cos(zy'/a + \alpha(z/a)), \tag{11.8}$$

with first-order poles at $z = \pm i/b_j$. By making use of the definition for $\alpha(k)$ we see that $e^{2i\alpha(0)} = -1$, and hence $\cos(2\alpha(0)) = -1$, which implies that $f(0) = 0$. The resulting Wightman function from (11.5) is found to be

$$\begin{aligned} W(x,x') &= W_0(x,x') + \frac{4}{(2\pi)^D} \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_{k_m}^\infty du \\ &\quad \times \frac{\cosh(yu + \tilde{\alpha}(u))\cosh(y'u + \tilde{\alpha}(u))}{\frac{(\beta_1 u - 1)(\beta_2 u - 1)}{(\beta_1 t + 1)(\beta_2 t + 1)} e^{2au} - 1} \frac{\cosh\left[\Delta t\sqrt{u^2 - k_m^2}\right]}{\sqrt{u^2 - k_m^2}}, \end{aligned} \tag{11.9}$$

where $k_m = \sqrt{k^2 + m^2}$ and the function $\tilde{\alpha}_j(t)$ is defined by the relation

$$e^{2\tilde{\alpha}(u)} \equiv \frac{\beta_1 u - 1}{\beta_1 u + 1}. \tag{11.10}$$

In formula (11.9),

$$\begin{aligned} W_0(x,x') &= W_M(x,x') + \int \frac{d\mathbf{k}}{(2\pi)^D} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_0^\infty du \frac{e^{-i\omega(u)\Delta t}}{\omega(u)} \cos\left[u(y + y') + 2\alpha(u)\right] \\ &\quad + \frac{\theta(\beta_1) e^{-(y+y')/\beta_1}}{(2\pi)^{D-1}\beta_1} \int d\mathbf{k} \frac{\exp(i\mathbf{k}\cdot\Delta\mathbf{x} - i\Delta t\sqrt{k_m^2 - 1/\beta_1^2})}{\sqrt{k_m^2 - 1/\beta_1^2}}, \end{aligned} \tag{11.11}$$

is the Wightman function for a single plate located at $y=0$ and $W_M(x,x')$ is the Wightman function in the Minkowski spacetime without boundaries. The last term on the right comes from the bound state present in the case $\beta_1>0$. To escape the instability of the vacuum state, we will assume that $m\beta_1 \geqslant 1$. On taking the coincidence limit, for the VEV of the field square we obtain the formula

$$\langle\varphi^2\rangle_{\mathrm{R,ren}} = \langle\varphi^2\rangle^{(0)}_{\mathrm{R,ren}} + 4\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty dt \frac{(t^2-m^2)^{D/2-1}}{\frac{(\beta_1 t-1)(\beta_2 t-1)}{(\beta_1 t+1)(\beta_2 t+1)}e^{2at}-1}\cosh^2(ty+\tilde{\alpha}(t)), \tag{11.12}$$

where

$$\langle\varphi^2\rangle^{(0)}_{\mathrm{R,ren}} = \frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty dt\,(t^2-m^2)^{D/2-1}e^{-2ty}\frac{\beta_1 t+1}{\beta_1 t-1}, \tag{11.13}$$

is the corresponding VEV in the region $y>0$ for a single plate at $y=0$. The surface divergences on the plate at $y=0$ are contained in this term. The second term on the right of formula (11.12) is finite at $y=0$ and is induced by the second plate located at $y=a$. This term diverges at $y=a$. The corresponding divergence is the same as that for the geometry of a single plate located at $y=a$. Note that in obtaining (11.13) from (11.11) we have written the cos function in the second integral term on the right of (11.11) as a sum of exponentials and have rotated the integration contour by the angle $\pi/2$ and by $-\pi/2$ for separate exponentials. For $\beta_1>0$ the corresponding integrals have poles $\pm i/\beta_1$ on the imaginary axis and the contribution from these poles cancel the part coming from the last term on the right of (11.11).

The VEV of the EMT is evaluated by formula (9.10). By taking into account formulae (11.9), (11.12), for the region between the plates one finds (no summation over l, see [32] for the case of a massless field)

$$\langle T_i^l\rangle_{\mathrm{R,ren}} = \langle T_i^l\rangle^{(0)}_{\mathrm{R,ren}} + 2\delta_i^l\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty dt\,\frac{(t^2-m^2)^{D/2-1}}{\frac{(\beta_1 t-1)(\beta_2 t-1)}{(\beta_1 t+1)(\beta_2 t+1)}e^{2at}-1}f_l(t,y), \tag{11.14}$$

where

$$f_l(t,x) = \left[4t^2(\zeta_D-\zeta)+m^2/D\right]\cosh(2ty+2\tilde{\alpha}(t)) - (t^2-m^2)/D, \tag{11.15}$$

for $l=0,1,\ldots,D-1$, and $f_D(t,y)=t^2$. In formula (11.14),

$$\begin{aligned}\langle T_i^l\rangle^{(0)}_{\mathrm{R,ren}} \quad &= \quad \delta_i^l\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\int_m^\infty dt(t^2-m^2)^{D/2-1}e^{-2yt} \\ &\times\frac{\beta_1 t+1}{\beta_1 t-1}\left[4(\zeta_D-\zeta)t^2+m^2/D\right], \end{aligned}\tag{11.16}$$

for $i=0,1,\ldots,D-1$, and $\langle T_D^D\rangle^{(0)}_{\mathrm{R,ren}}=0$, are the VEVs in the region $y>0$ induced by a single plate at $y=0$, and the second term on the right is the part of the energy-momentum tensor induced by the presence of the second plate. In [50], formulae (11.12) and (11.14) are obtained as limiting cases of the corresponding results for the geometry of two parallel plates uniformly accelerated through the Fulling-Rindler vacuum (see below). For a conformally coupled massless scalar field the vacuum EMT is uniform and traceless. Note that we have investigated the vacuum densities in the bulk. For Robin boundary conditions in addition to this part there is a contribution to the EMT located on the plates.

Vacuum forces acting on the plates are determined by $\langle T_D^D\rangle_{\mathrm{R,ren}}$. This component is uniform and, hence, is finite on the plates. The latter property is a consequence of the high symmetry of the problem and is not valid for curved boundaries. In dependence of the values for the coefficients β_j the vacuum forces can be both attractive or repulsive. In particular, the vacuum

forces are repulsive in the case of Dirichlet boundary condition on one plate and Neumann boundary condition on the other. In the case of a massless scalar field for the modulus of the corresponding force from (11.14) with $i = l = D$ one finds $F_{\mathrm{DN}} = (1 - 2^{-D})F_{\mathrm{D}}$, where F_{D} is given by formula (10.17). Taking $D = 3$ and summing over the polarization states, from here we obtain the result of [96] for the electromagnetic Casimir force between two parallel plates one of which is a perfect conductor and the other is infinitely permeable.

Series over the zeros of the function $F(z)$ from (11.3) appear also related to the Casimir effect for two parallel plates with non-local boundary conditions

$$n^{\nu}_{(j)}\partial_{\nu}\varphi(x^{\mu}) + \int d\mathbf{x}'_{\parallel}\, f_j(|\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}|)\varphi(x'^{\mu}) = 0,\ y = a_j, \tag{11.17}$$

where $n^{\nu}_{(j)}$ is the inward-pointing unit normal to the boundary at $y = a_j$. In the region between the plates the corresponding eigenfunctions have the form (11.1) and the eigenvalues for k are zeros of the function $F(ka)$ in (11.3), where now the coefficients are given by the formula $1/b_j = (-1)^{j-1}aF_j(k_{\parallel})$ with

$$F_j(k_{\parallel}) \equiv \int d\mathbf{x}_{\parallel}\, f_j(|\mathbf{x}_{\parallel}|)e^{i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} = \frac{(2\pi)^{(D-1)/2}}{k_{\parallel}^{(D-3)/2}} \int_0^{\infty} du\, u^{(D-1)/2} f_j(u) J_{(D-3)/2}(uk_{\parallel}). \tag{11.18}$$

The corresponding VEVs are investigated in [33].

12 Scalar vacuum densities for spherical boundaries in the global monopole bulk

Historically, the investigation of the Casimir effect for a spherical shell was motivated by the Casimir semiclassical model of an electron. In this model Casimir suggested that Poincare stress, to stabilize the charged particle, could arise from vacuum quantum fluctuations and the fine structure constant can be determined by a balance between the Casimir force (assumed attractive) and the Coulomb repulsion. However, as it has been shown by Boyer [97], the electromagnetic Casimir energy for the perfectly conducting sphere is positive, implying a repulsive force. This result has later been reconsidered by a number of authors [98, 81, 99]. More recently new methods have been developed for this problem including direct mode summation techniques [100, 101, 102] (see also [3, 6, 7] and references therein). However, the main part of the studies was focused on global quantities such as the total energy and the vacuum forces acting on the sphere. The investigation of the energy distribution inside a perfectly reflecting spherical shell was made in [103] in the case of QED and in [104] for QCD. The distribution of the other components for the electromagnetic EMT inside as well as outside the shell can be obtained from the results of [105, 106]. In these papers consideration was carried out in terms of Schwinger's source theory. In [34, 35, 36] the calculations of the the VEVs for the EMT components inside and outside the perfectly conducting spherical shell are based on the GAPF. The same method was used in [38] for the evaluation of the Wightman function and the VEVs for the EMT of a massive scalar field with general curvature coupling and obeying Robin boundary condition on spherically symmetric boundaries in D-dimensional Minkowski space. In [107] the local VEVs outside a spherical boundary and the corresponding energy conditions are investigated using the calculational framework developed in [108]. More complicated problems with spherical boundaries in the background spacetime of a global monopole are considered in [51, 52, 53, 54, 55] for both scalar and fermionic fields. As the corresponding results in the Minkowski bulk are obtained as special cases, here we discuss the application of the GAPF for the geometry of spherical boundaries in the global monopole bulk.

Global monopoles are spherically symmetric topological defects created due to phase transition when a global symmetry is spontaneously broken and they have important role in cosmology and astrophysics. The global monopole was first introduced by Sokolov and Starobinsky [109]. A few years later, the gravitational effects of the global monopole were considered in [110], where a solution is presented which describes a global monopole at large radial distances. Neglecting the small size of the monopole core, in the hyperspherical polar coordinates $(r, \vartheta, \phi) \equiv (r, \theta_1, \theta_2, \ldots \theta_n, \phi)$, $n = D - 2$, the corresponding metric tensor generalized to an arbitrary number of spatial dimensions can be approximately given by the line element

$$ds^2 = dt^2 - dr^2 - \alpha^2 r^2 d\Omega_D^2, \tag{12.1}$$

where $d\Omega_D^2$ is the line element on a surface of the unit sphere in D-dimensional Euclidean space, the parameter α is related to the symmetry breaking energy scale in the theory. The solid angle corresponding to (12.1) is $\alpha^2 S_D$ with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ being the total area of the surface of the unit sphere in D-dimensional Euclidean space. The nonzero components of the Ricci tensor and Ricci scalar for the metric corresponding to line element (12.1) are given by expressions (no summation over i)

$$R_i^i = n\frac{1-\alpha^2}{\alpha^2 r^2}, \quad R = n(n+1)\frac{1-\alpha^2}{\alpha^2 r^2}, \tag{12.2}$$

where the indices $i = 2, 3, \ldots, D$ correspond to the coordinates $\theta_1, \theta_2, \ldots, \phi$, respectively. Note that for $\alpha \neq 1$ the geometry is singular at the origin (point-like monopole, for vacuum polarization effects by a global monopole with finite core see [111]). In this section we will consider the applications of the GAPF for the evaluation of the Wightman function, VEVs of the field square and the EMT for a scalar field with general curvature coupling parameter under the presence of spherical boundaries concentric with the global monopole.

12.1 Region inside a single sphere

For the region inside a sphere with radius a the complete set of solutions to (9.1) is specified by the set of quantum numbers $\sigma = (\lambda, m_k)$, where $m_k = (m_0 \equiv l, m_1, \ldots, m_n)$ and $m_1, m_2, \ldots, m_n$ are integers such that $0 \leqslant m_{n-1} \leqslant m_{n-2} \leqslant \cdots \leqslant m_1 \leqslant l$, $-m_{n-1} \leqslant m_n \leqslant m_{n-1}$. The corresponding eigenfunctions have the form

$$\varphi_\sigma(x) = \beta_\sigma r^{-n/2} J_{\nu_l}(\lambda r) Y(m_k; \vartheta, \phi) e^{-i\omega t}, \; \omega = \sqrt{\lambda^2 + m^2}, \; l = 0, 1, 2, \ldots, \tag{12.3}$$

where the order of the Bessel function is given by the formula

$$\nu_l = \frac{1}{\alpha}\left[\left(l + \frac{n}{2}\right)^2 + (1-\alpha^2)n\left((n+1)\zeta - \frac{n}{4}\right)\right]^{1/2}, \tag{12.4}$$

and $Y(m_k; \vartheta, \phi)$ is the hyperspherical harmonic of degree l. The coefficient β_σ is found from the normalization condition and is equal to

$$\beta_\sigma^2 = \frac{\lambda T_{\nu_l}(\lambda a)}{N(m_k)\omega a \alpha^{D-1}}, \tag{12.5}$$

with $T_\nu(z)$ defined by (3.8). From boundary condition (9.11) for eigenfunctions (12.3), we see that the eigenvalues for λa are zeros of the function $\bar{J}_{\nu_l}(\lambda a)$ with the barred notation from (3.1) and with the coefficients

$$A = \tilde{A} - nB/2, \quad B = n^1 \tilde{B}/a, \tag{12.6}$$

where $n^1 = -1$ for the region inside the sphere. In the notations of Section 3, for the eigenvalues of λ and ω one finds

$$\lambda = \lambda_{\nu_l,k}/a,\ \omega = \sqrt{\lambda_{\nu_l,k}^2/a^2 + m^2}. \tag{12.7}$$

Substituting the functions (12.3) into (9.8) and using the addition formula for the hyperspherical harmonics [13], for the Wightman function we obtain

$$\begin{aligned} W(x,x') &= \frac{(rr')^{-n/2}}{naS_D\alpha^{D-1}} \sum_{l=0}^{\infty}(2l+n)C_l^{n/2}(\cos\theta) \\ &\times \sum_{k=1}^{\infty} \frac{\lambda_{\nu_l,k}T_{\nu_l}(\lambda_{\nu_l,k})}{\sqrt{\lambda_{\nu_l,k}^2 + m^2a^2}} J_{\nu_l}(\lambda_{\nu_l,k}r/a)J_{\nu_l}(\lambda_{\nu_l,k}r'/a)e^{-i\omega_{\nu_l,k}\Delta t}, \end{aligned} \tag{12.8}$$

where $\Delta t = t - t'$. In this formula, $C_p^q(x)$ is the Gegenbauer polynomial of degree p and order q, and θ is the angle between the directions (ϑ, ϕ) and (ϑ', ϕ'). To sum over k we will use summation formula (3.18), taking

$$f(z) = zJ_{\nu_l}(zr/a)J_{\nu_l}(zr'/a)e^{-i\Delta t\sqrt{z^2/a^2+m^2}}. \tag{12.9}$$

For $|z|/a < m$ this function satisfies the relation $f(ze^{-\pi i/2}) = -e^{-2\nu\pi i}f(ze^{\pi i/2})$. As a result, the integral on the right of formula (3.18) over the interval $(0, ma)$ vanishes. Conditions (3.4) are satisfied if $r + r' + |\Delta t| < 2a$. After the application of formula (3.18) the Wightman function is presented in the form [51]

$$W^{(a)}(x,x') = W_{\mathrm{m}}(x,x') + \langle\varphi(x)\varphi(x')\rangle_{\mathrm{b}}, \tag{12.10}$$

where

$$W_{\mathrm{m}}(x,x') = \frac{\alpha^{1-D}}{2nS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \int_0^{\infty} dz\, \frac{ze^{-i\Delta t\sqrt{z^2+m^2}}}{\sqrt{z^2+m^2}} J_{\nu_l}(zr)J_{\nu_l}(zr'), \tag{12.11}$$

and

$$\begin{aligned} \langle\varphi(x)\varphi(x')\rangle_{\mathrm{b}} &= -\frac{\alpha^{1-D}}{\pi nS_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \int_m^{\infty} dz\, z \frac{\bar{K}_{\nu_l}(az)}{\bar{I}_{\nu_l}(az)} \\ &\times \frac{I_{\nu_l}(zr)I_{\nu_l}(zr')}{\sqrt{z^2-m^2}} \cosh(\Delta t\sqrt{z^2-m^2}). \end{aligned} \tag{12.12}$$

The contribution of the term (12.11) to the VEV does not depend on the sphere radius, whereas the contribution of the term (12.12) vanishes in the limit $a \to \infty$. It follows from here that expression (12.11) is the Wightman function in the unbounded global monopole space (for the corresponding Euclidean Green function see [112, 113]). This can also be seen by explicit evaluation of the mode-sum using the eigenfunctions for the global monopole spacetime without boundaries. As we see, the application of the GAPF allows to extract from the bilinear field product the contribution due to the unbounded monopole spacetime, and the term (12.12) can be interpreted as the part of the VEV induced by the spherical boundary.

The VEV of the field square is obtained by using formula (9.9). For $0 < r < a$ the divergences are contained in the first summand on the right of Eq. (12.10) only. Hence, the renormalization procedure for the local characteristics of the vacuum, such as field square and EMT, is the same as for the global monopole geometry without boundaries. This procedure is discussed in a number of papers [112, 113, 114] and is a useful illustration for the general renormalization

prescription on curved backgrounds. Here we are interested in the parts of the VEVs induced by the presence of a spherical shell. Using the relation $C_l^{n/2}(1) = \Gamma(l+n)/\Gamma(n)l!$, for the corresponding boundary part in the VEV of the field square we get

$$\langle\varphi^2\rangle_{\rm b} = -\frac{\alpha^{1-D}}{\pi r^n S_D}\sum_{l=0}^{\infty} D_l \int_m^{\infty} dz\ \frac{\bar{K}_{\nu_l}(az)}{\bar{I}_{\nu_l}(az)}\frac{zI_{\nu_l}^2(rz)}{\sqrt{z^2-m^2}}, \tag{12.13}$$

where

$$D_l = (2l+D-2)\frac{\Gamma(l+D-2)}{\Gamma(D-1)\,l!} \tag{12.14}$$

is the degeneracy of each angular mode with given l. As it has been noted before, expression (12.13) is finite for all values $0 < r < a$. For a given l and large z the integrand behaves as $e^{2z(r/a-1)}/z$ and, hence, the integral converges when $r < a$. For large values l, introducing a new integration variable $y = z/\nu_l$ in the integral of Eq. (12.13) and using the uniform asymptotic expansions for the modified Bessel functions [65], it can be seen that the both integral and sum are convergent for $r < a$ and diverge at $r = a$. For points near the sphere the leading term of the corresponding asymptotic expansion over the distance from the boundary has the form

$$\langle\varphi^2\rangle_{\rm b} \approx \frac{(1-2\delta_{B0})\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{(D+1)/2}(a-r)^{D-1}}. \tag{12.15}$$

This term does not depend on the mass and on the parameter α and is the same as that for a sphere on the Minkowski bulk. As the purely gravitational part is finite for $r = a$, we see that near the sphere surface the VEV of the field square is dominated by the boundary-induced part. The boundary-induced VEV (12.13) is zero at the sphere center for $(\alpha^{-2}-1)\zeta > 0$, non-zero constant for $(\alpha^{-2}-1)\zeta = 0$, and is infinite for $(\alpha^{-2}-1)\zeta < 0$.

Substituting the Wightman function (12.10) into Eq. (9.10), for the VEV of the EMT inside the spherical shell one finds [51]

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle_{\rm m} + \langle T_i^k\rangle_{\rm b}, \tag{12.16}$$

where $\langle T_i^k\rangle_{\rm m}$ is the corresponding VEV for the monopole geometry when the boundary is absent, and the part $\langle T_i^k\rangle_{\rm b}$ is induced by the presence of the spherical shell. From Eqs. (12.11), (12.12), and (12.13), for the boundary-induced parts of the EMT components one obtains (no summation over i)

$$\langle T_i^k\rangle_{\rm b} = -\frac{\alpha^{1-D}\delta_i^k}{2\pi r^n S_D}\sum_{l=0}^{\infty} D_l \int_m^{\infty} dz\, z^3\frac{\bar{K}_{\nu_l}(za)}{\bar{I}_{\nu_l}(za)}\frac{F_{\nu_l}^{(i)}\left[I_{\nu_l}(zr)\right]}{\sqrt{z^2-m^2}}, \quad r<a, \tag{12.17}$$

where for a given function $f(y)$ we have introduced the notations

$$\begin{aligned} F_{\nu_l}^{(0)}\left[f(y)\right] &= (1-4\zeta)\left[f'^2(y) - \frac{n}{y}f(y)f'(y) \right. \\ &\quad \left. + \left(\frac{\nu_l^2}{y^2} - \frac{1+4\zeta-2(mr/y)^2}{1-4\zeta}\right)f^2(y)\right], \end{aligned} \tag{12.18}$$

$$F_{\nu_l}^{(1)}\left[f(y)\right] = f'^2(y) + \frac{\tilde{\zeta}}{y}f(y)f'(y) - \left(1+\frac{\nu_l^2+\tilde{\zeta}n/2}{y^2}\right)f^2(y), \tag{12.19}$$

$$\begin{aligned} F_{\nu_l}^{(i)}\left[f(y)\right] &= (4\zeta-1)f'^2(y) - \frac{\tilde{\zeta}}{y}f(y)f'(y) \\ &\quad + \left[4\zeta-1+\frac{\nu_l^2(1+\tilde{\zeta})+\tilde{\zeta}n/2}{(n+1)y^2}\right]f^2(y), \end{aligned} \tag{12.20}$$

with $\tilde{\zeta} = 4(n+1)\zeta - n$ and in (12.20) $i = 2, 3, \ldots, D$. It can be seen that components (12.17) satisfy the continuity equation $\nabla_k \langle T_i^k \rangle_{\rm b} = 0$ and are finite for $0 < r < a$. They are zero at the sphere center for $n\zeta(\alpha^{-2} - 1) > 1$, are non-zero constants for $n\zeta(\alpha^{-2} - 1) = 1$, and are infinite otherwise. These singularities appear because the geometrical characteristics of global monopole spacetime are divergent at the origin. However, note that the corresponding contribution to the total energy of the vacuum inside a sphere coming from $\langle T_0^0 \rangle_{\rm b}$ is finite due to the factor r^{D-1} in the volume element.

Expectation values (12.17) diverge at the sphere surface, $r \to a$. The corresponding asymptotic behavior can be found using the uniform asymptotic expansions for the modified Bessel functions, and the leading terms are determined by the relations

$$\langle T_0^0 \rangle_{\rm b} \sim \langle T_2^2 \rangle_{\rm b} \sim \frac{Da\langle T_1^1 \rangle_{\rm b}}{(D-1)(a-r)} \sim -\frac{D\Gamma((D+1)/2)(\zeta - \zeta_D)}{2^D \pi^{(D+1)/2}(a-r)^{D+1}} (1 - 2\delta_{B0}) . \tag{12.21}$$

As for the field square, these terms do not depend on mass and parameter α, and are the same as for a spherical shell in the Minkowski bulk. In general, asymptotic series can be developed in powers of the distance from the boundary. The corresponding subleading coefficients will depend on the mass, Robin coefficient, and parameter α. Due to the surface divergencies near the sphere surface the total vacuum EMT is dominated by the boundary-induced part $\langle T_i^k \rangle_{\rm b}$.

12.2 Vacuum densities in the region between two spheres

In this subsection we consider the VEVs of the field square and the EMT on background of the geometry described by (12.1), assuming that the field satisfies Robin boundary condition (9.11) on two concentric spherical boundaries with radii a and b, $a < b$. We will consider the general case when the coefficients in the boundary conditions for the inner and outer spheres are different and will denote them by $\tilde{A}_j$ and $\tilde{B}_j$ with $j = a, b$. For the region between two spheres the corresponding eigenfunctions can be obtained from formula (12.3) with the replacement

$$J_{\nu_l}(\lambda r) \to g_{\nu_l}(\lambda a, \lambda r) \equiv J_{\nu_l}(\lambda r)\bar{Y}_{\nu_l}^{(a)}(\lambda a) - \bar{J}_{\nu_l}^{(a)}(\lambda a) Y_{\nu_l}(\lambda r), \tag{12.22}$$

where the functions with overbars are defined by (4.2) with

$$A_j = \tilde{A}_j - B_j n/2, \quad B_j = n_j \tilde{B}_j / j, \ j = a, b, \ n_a = 1, \ n_b = -1. \tag{12.23}$$

Substituting eigenfunctions (12.3) into the normalization condition and using the value for the standard integral involving the square of a cylinder function [66], one finds

$$\beta_\sigma^2 = \frac{\pi^2 \lambda T_{\nu_l}^{ab}(b/a, \lambda a)}{4N(m_k)\omega a \alpha^{D-1}}, \tag{12.24}$$

where $T_\nu^{ab}(\eta, z)$ is defined by (4.7).

The functions chosen in the form (12.22) satisfy the boundary condition on the inner sphere. From the boundary condition at $r = b$ one obtains that the corresponding eigenmodes are solutions to the equation

$$C_{\nu_l}^{ab}(b/a, \lambda a) = 0, \tag{12.25}$$

with the function $C_\nu^{ab}(\eta, z)$ defined by formula (4.1). Hence, in the region between two spheres the eigenvalues for λ and ω are

$$\lambda = \gamma_{\nu_l,k}/a, \ \omega = \sqrt{\gamma_{\nu_l,k}^2/a^2 + m^2}. \tag{12.26}$$

Substituting the eigenfunctions into mode-sum (9.8) and using the addition formula for the spherical harmonics, for the Wightman function one finds

$$W(x,x')=\frac{\pi^2(rr')^{-n/2}}{4naS_D\alpha^{D-1}}\sum_{l=0}^{\infty}(2l+n)C_l^{n/2}(\cos\theta)\sum_{k=1}^{\infty}h(\gamma_{\nu_l,k})T_{\nu_l}^{ab}(b/a,\gamma_{\nu_l,k}), \tag{12.27}$$

with the function

$$h(z)=\frac{ze^{-i\Delta t\sqrt{z^2/a^2+m^2}}}{\sqrt{z^2+m^2a^2}}g_{\nu_l}(z,zr/a)g_{\nu_l}(z,zr'/a). \tag{12.28}$$

To sum over k we will use summation formula (4.14). The corresponding conditions for function (12.28) are satisfied if $r+r'+|\Delta t|<2b$. In particular, this is the case in the coincidence limit for the region under consideration. For the Wightman function one obtains

$$\begin{aligned} W(x,x') &= \frac{\alpha^{1-D}}{2nS_D}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta) \\ &\times\left\{\frac{1}{a}\int_0^{\infty}\frac{h(z)dz}{\bar{J}_{\nu_l}^{(a)2}(z)+\bar{Y}_{\nu_l}^{(a)2}(z)}-\frac{2}{\pi}\int_m^{\infty}dz\,\frac{z\Omega_{a\nu}(az,bz)}{\sqrt{z^2-a^2}}\right. \\ &\left.\times G_{\nu_l}^{(a)}(az,zr)G_{\nu_l}^{(a)}(az,zr')\cosh(\Delta t\sqrt{z^2-m^2})\right\}, \end{aligned} \tag{12.29}$$

where we have introduced the notation

$$G_{\nu}^{(j)}(z,y)=I_{\nu}(y)\bar{K}_{\nu}^{(j)}(z)-\bar{I}_{\nu}^{(j)}(z)K_{\nu}(y),\ j=a,b, \tag{12.30}$$

and the function $\Omega_{a\nu}(az,bz)$ is defined by relation (4.15). In the limit $b\to\infty$ the second integral on the right of (12.29) tends to zero, whereas the first one does not depend on b. It follows from here that the term with the first integral in the figure braces corresponds to the Wightman function for the region outside a single sphere with radius a on background of the global monopole geometry. The latter we will denote by $W^{(a)}(x,x')$. So, the Wightman function in the region between two spheres is presented in the form [53]

$$\begin{aligned} W(x,x') &= W^{(a)}(x,x')-\frac{\alpha^{1-D}}{\pi nS_D}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta)\int_m^{\infty}dz\frac{z\Omega_{a\nu_l}(az,bz)}{\sqrt{z^2-m^2}} \\ &\times G_{\nu_l}^{(a)}(az,rz)G_{\nu_l}^{(a)}(az,r'z)\cosh(\Delta t\sqrt{z^2-m^2}). \end{aligned} \tag{12.31}$$

To extract from the function $W^{(a)}(x,x')$ the part induced by the presence of the sphere we use the identity

$$\frac{g_{\nu}(z,y)g_{\nu}(z,y')}{\bar{J}_{\nu}^2(z)+\bar{Y}_{\nu}^2(z)}=J_{\nu}(y)J_{\nu}(y')-\frac{1}{2}\sum_{s=1}^{2}\frac{\bar{J}_{\nu}(z)}{\bar{H}_{\nu}^{(s)}(z)}H_{\nu}^{(s)}(y)H_{\nu}^{(s)}(y'). \tag{12.32}$$

The contribution from the first term on the right of this relation with $y=xr/a$ and $y'=xr'/a$ gives the Wightman function $W_{\rm m}(x,x')$ for the geometry without boundaries. In the part coming from the second term we rotate the contour for the integration over z by the angle $\pi/2$ for $s=1$ and by the angle $-\pi/2$ for $s=2$. Introducing the modified Bessel functions, one finds

$$\begin{aligned} W^{(a)}(x,x') &= W_{\rm m}(x,x')-\frac{\alpha^{1-D}}{\pi nS_D}\sum_{l=0}^{\infty}\frac{2l+n}{(rr')^{n/2}}C_l^{n/2}(\cos\theta)\int_m^{\infty}dz\,z \\ &\times\frac{\bar{I}_{\nu_l}(az)}{\bar{K}_{\nu_l}(az)}\frac{K_{\nu_l}(zr)K_{\nu_l}(zr')}{\sqrt{z^2-m^2}}\cosh(\Delta t\sqrt{z^2-m^2}). \end{aligned} \tag{12.33}$$

Note that in the coincidence limit, $x' = x$, the second summand on the right hand-side of (12.31) will give a finite result for $a \leqslant r < b$, and is divergent on the boundary $r = b$. It can be seen that for the case of two spheres the Wightman function in the intermediate region can also be presented in the form

$$\begin{aligned} W(x,x') &= W^{(b)}(x,x') - \frac{\alpha^{1-D}}{\pi n S_D} \sum_{l=0}^{\infty} \frac{2l+n}{(rr')^{n/2}} C_l^{n/2}(\cos\theta) \\ &\times \int_m^{\infty} dz \frac{z\Omega_{b\nu_l}(az,bz)}{\sqrt{z^2-m^2}} G_{\nu_l}^{(b)}(bz,rz) G_{\nu_l}^{(b)}(bz,r'z) \cosh(\Delta t\sqrt{z^2-m^2}), \end{aligned} \tag{12.34}$$

with $W^{(b)}(x,x')$ being the Wightman function for the vacuum inside a single sphere with radius b, and

$$\Omega_{b\nu}(x,\lambda x) = \frac{\bar{I}_\nu^{(a)}(x)/\bar{I}_\nu^{(b)}(\lambda x)}{\bar{K}_\nu^{(a)}(x)\bar{I}_\nu^{(b)}(\lambda x) - \bar{K}_\nu^{(b)}(\lambda x)\bar{I}_\nu^{(a)}(x)}. \tag{12.35}$$

In the coincidence limit, the second summand on the right of formula (12.34) is finite for $a < r \leqslant b$ and diverges on the boundary $r = a$.

The VEV of the field square is obtained computing the Wightman function in the coincidence limit $x' \to x$. In the region between the spheres, from (12.31) and (12.34) we obtain two equivalent forms [53]

$$\langle 0|\varphi^2|0\rangle = \langle \varphi^2\rangle^{(j)} - \frac{\alpha^{1-D}}{\pi n S_D r^n} \sum_{l=0}^{\infty} D_l \int_m^{\infty} dz \frac{z\Omega_{j\nu_l}(az,bz)}{\sqrt{z^2-m^2}} G_{\nu_l}^{(j)2}(jz,rz), \tag{12.36}$$

with $j = a, b$, and $\langle \varphi^2\rangle^{(j)}$ is the VEV for the case of a single sphere with radius j.

Using the Wightman function from (12.31) and the VEV for the field square from (12.36), the components of the vacuum EMT can be presented in two equivalent forms corresponding to $j = a, b$ (no summation over i) [53]:

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle^{(j)} - \frac{\alpha^{1-D}\delta_i^k}{2\pi r^n S_D} \sum_{l=0}^{\infty} D_l \int_m^{\infty} dz \frac{z^3\Omega_{j\nu_l}(az,bz)}{\sqrt{z^2-m^2}} F_{\nu_l}^{(i)}[G_{\nu_l}^{(j)}(jz,rz)]. \tag{12.37}$$

Here $\langle T_i^k\rangle^{(j)}$ is the vacuum EMT for the case of a single sphere with radius j and the functions $F_\nu^{(i)}[f(y)]$ are defined by relations (12.18)-(12.20). In the region outside a single sphere the expressions for the boundary-induced parts are obtained from the corresponding formula for the interior region given in the previous subsection by the replacements $I \rightleftarrows K$. The scalar vacuum densities for spherical boundaries in the Minkowski spacetime (see [38]) are obtained from the formulae in this section as a special case taking $\alpha = 1$. Note that in this case $\nu_l = l + n/2$.

The vacuum force per unit surface of the sphere at $r = j$ is determined by the $^1_1$ – component of the vacuum EMT at this point. For the region between two spheres the corresponding effective pressures can be presented as a sum of two terms:

$$p^{(j)} = p_1^{(j)} + p_{(\mathrm{int})}^{(j)}, \quad j = a, b. \tag{12.38}$$

The first term on the right is the pressure for a single sphere at $r = j$ when the second sphere is absent. This term is divergent due to the surface divergences in the VEVs of local physical observables. Here we will be concerned with the second term on the right of Eq. (12.38) which is the pressure induced by the presence of the second sphere, and can be termed as an interaction force. Unlike to the single shell parts, this term is free from renormalization ambiguities and is

determined by the last term on the right of formula (12.37). From the formula for the EMT, for the interaction parts of the vacuum forces per unit surface one finds [53]

$$
\begin{aligned}
p^{(j)}_{(\mathrm{int})} &= \frac{n_j \alpha^{1-D}}{2\pi j^{D-1} S_D} \sum_{l=0}^{\infty} D_l \int_m^\infty \frac{z dz}{\sqrt{z^2-m^2}} \left[1 - \frac{\tilde{\zeta} \tilde{A}_j B_j}{A_j^2 - B_j^2 (z^2 j^2 + \nu_l^2)} \right] \\
&\times \frac{\partial}{\partial j} \ln \left| 1 - \frac{\bar{I}^{(a)}_{\nu_l}(az)}{\bar{I}^{(b)}_{\nu_l}(bz)} \frac{\bar{K}^{(b)}_{\nu_l}(bz)}{\bar{K}^{(a)}_{\nu_l}(az)} \right|, \qquad j = a, b, \qquad (12.39)
\end{aligned}
$$

where $\tilde{\zeta}$ is defined after formula (12.20). The expression on the right of this formula is finite for all non-zero distances between the shells. For Dirichlet and Neumann scalars the second term in the square brackets is zero. In the case of Dirichlet scalar the interaction forces are always attractive. For the general Robin case the interaction force can be either attractive or repulsive in dependence on the coefficients in the boundary conditions.

13 Casimir effect for a fermionic field in a global monopole background with spherical boundaries

13.1 Vacuum energy-momentum tensor inside a spherical shell

Our interest in this section will be the VEV of the EMT for a fermionic field with mass M induced by spherical boundaries in the global monopole spacetime. The dynamics of the field on a curved spacetime is governed by the Dirac equation

$$
i\gamma^l (\partial_l + \Gamma_l)\psi - M\psi = 0 \ , \qquad (13.1)
$$

where γ^l are the Dirac matrices and $\Gamma_l = \gamma_k \nabla_l \gamma^k / 4$ is the spin connection with ∇_l being the standard covariant derivative operator. We will assume that the field satisfies bag boundary condition

$$
\left(1 + i\gamma^l n_{\mathrm{b}l} \right) \psi = 0 \ , \quad r = a, \qquad (13.2)
$$

on the sphere with radius a, $n_{\mathrm{b}l}$ is the outward-pointing normal to the boundary. Expanding the field operator in terms of a complete set of single-particle states $\{\psi^{(+)}_\beta, \psi^{(-)}_\beta\}$ and making use of the standard anticommutation relations, for the VEV of the EMT one finds the following mode-sum formula

$$
\langle 0 | T_{ik} | 0 \rangle = \sum_\beta T_{ik} \{ \bar{\psi}^{(-)}_\beta(x), \psi^{(-)}_\beta(x) \}. \qquad (13.3)
$$

For the geometry under consideration the eigenfunctions are specified by the set of quantum numbers $\beta = (kjm\sigma)$, where $j = 1/2, 3/2, \ldots$ determines the value of the total angular momentum, $m = -j, \ldots, j$ is its projection, and $\sigma = 0, 1$ corresponds to two types of eigenfunctions with different parities. These functions have the form

$$
\psi^{(\pm)}_\beta = A_\sigma \frac{e^{-i\omega t}}{\sqrt{r}} \begin{pmatrix} Z_{\nu_\sigma}(kr) \Omega_{jl_\sigma m}(\theta, \varphi) \\ i n_\sigma Z_{\nu_\sigma + n_\sigma}(kr) \frac{k(\hat{n}\cdot\vec{\sigma})}{\omega + M} \Omega_{jl_\sigma m}(\theta, \varphi) \end{pmatrix}, \quad n_\sigma = (-1)^\sigma, \qquad (13.4)
$$

$$
\omega = \pm E, \quad E = \sqrt{k^2 + M^2} \ , \quad \nu_\sigma = \frac{j + 1/2}{\alpha} - \frac{n_\sigma}{2} \qquad (13.5)
$$

where $\hat{n} = \vec{r}/r$, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ with the curved space Pauli matrices σ^k. In Eq. (13.4), $Z_\nu(x)$ represents a cylinder function of the order ν and $\Omega_{jl_\sigma m}(\theta, \varphi)$ are the standard spinor spherical harmonics with $l_\sigma = j - n_\sigma/2$. As in mode-sum formula (13.3) the negative frequency modes

are employed, in the discussion below we will consider the eigenfunctions (13.4) with the lower sign.

For the region inside a spherical shell one has $Z_\nu(x) = J_\nu(x)$. The imposition of the boundary condition on the eigenfunctions (13.4) leads to the following equations for the eigenvalues

$$\tilde{J}_{\nu_\sigma}(ka) = 0 \ , \tag{13.6}$$

where for a given function $F(z)$ we use the notation

$$\tilde{F}(z) \equiv zF'(z) + (\mu_a - \sqrt{z^2+\mu_a^2} - n_\sigma \nu)F(z) \ , \quad \sigma = 0,\ 1, \tag{13.7}$$

with $\mu_a = Ma$. Let us denote by $\lambda^{\rm f}_{\nu_\sigma,s} = ka$, $s = 1, 2, \ldots$, the roots to equation (13.6) in the right half-plane, arranged in ascending order. By using the standard integral for the Bessel functions, for the normalization coefficient one finds

$$A_\sigma^2 = \frac{z}{2\alpha^2 a^2} \frac{\sqrt{z^2+\mu_a^2}+\mu_a}{\sqrt{z^2+\mu_a^2}} T^{\rm f}_{\nu_\sigma}(z) \ , \quad z = \lambda^{\rm f}_{\nu_\sigma,s} \ , \tag{13.8}$$

with

$$T^{\rm f}_\nu(z) = \frac{z}{J_\nu^2(z)} \left[z^2 + (\mu - n_\sigma \nu)(\mu_a - \sqrt{z^2+\mu_a^2}) + \frac{z^2}{2\sqrt{z^2+\mu_a^2}} \right]^{-1} . \tag{13.9}$$

Substituting eigenfunctions (13.4) into Eq. (13.3), the summation over the quantum number m can be done by using the standard summation formula for the spherical harmonics. For the EMT components one finds (no summation over i)

$$\langle 0|T_i^i|0\rangle = -\sum_{j=1/2}^{\infty} \frac{2j+1}{8\pi\alpha^2 a^3 r} \sum_{\sigma=0,1} \sum_{s=1}^{\infty} T^{\rm f}_{\nu_\sigma}(z) f^{(i)}_{\sigma\nu_\sigma}\left[z, J_{\nu_\sigma}(zr/a)\right]_{z=\lambda^{\rm f}_{\nu_\sigma,s}} , \tag{13.10}$$

where we have introduced the notations

$$f^{(0)}_{\sigma\nu}\left[z, g_\nu(y)\right] = z\left[(\sqrt{z^2+\mu_a^2}-\mu_a)g_\nu^2(y) + (\sqrt{z^2+\mu_a^2}+\mu_a)g^2_{\nu+n_\sigma}(y)\right], \tag{13.11}$$

$$f^{(1)}_{\sigma\nu}\left[z, g_\nu(y)\right] = \frac{z^3}{\sqrt{z^2+\mu_a^2}} \left[g_\nu^2(y) + g^2_{\nu+n_\sigma}(y) - \frac{2\nu+n_\sigma}{y} g_\nu(y) g_{\nu+n_\sigma}(y)\right], \tag{13.12}$$

$$f^{(i)}_{\sigma\nu}\left[z, g_\nu(y)\right] = \frac{z^3(2\nu+n_\sigma)}{2y\sqrt{z^2+\mu_a^2}} g_\nu(y) g_{\nu+n_\sigma}(y) \ , \ i = 2, 3. \tag{13.13}$$

In order to obtain a summation formula for series over the zeros $\lambda^{\rm f}_{\nu,s}$, in the GAPF as a function $g(z)$ we choose

$$g(z) = i\frac{\tilde{Y}_\nu(z)}{\tilde{J}_\nu(z)} f(z) \ , \tag{13.14}$$

with a function $f(z)$ analytic in the right half-plane ${\rm Re}\, z > 0$. By making use of the asymptotic formulae for the Bessel functions for large values of the argument, the conditions for the GAPF can be written in terms of the function $f(z)$ as follows:

$$|f(z)| < \epsilon(x) e^{c|y|} \ , \quad z = x + iy, \quad |z| \to \infty \ , \tag{13.15}$$

where $c < 2$ and $\epsilon(x) \to 0$ for $x \to \infty$. By using the Wronskian relation for the Bessel functions, one can see that $\tilde{Y}(\lambda^{\rm f}_{\nu_\sigma,s}) = 2/[\pi \tilde{J}(\lambda^{\rm f}_{\nu_\sigma,s})]$. This allows to present the residue term coming from the poles of the function $g(z)$ in the form

$$\pi i \mathop{\rm Res}_{z=\lambda^{\rm f}_{\nu,s}} g(z) = T^{\rm f}_\nu(\lambda^{\rm f}_{\nu,s}) f(\lambda^{\rm f}_{\nu,s}). \tag{13.16}$$

Substituting (13.14) and (13.16) into the GAPF (2.11) and taking in this formula the limit $a \to 0$ (the branch points $z = \pm i\mu_a$ are avoided by semicircles of small radius), we obtain that for a function $f(z)$ analytic in the half-plane $\mathrm{Re}\, z > 0$ and satisfying condition (13.15) the following formula takes place [54]

$$\lim_{b\to+\infty}\left[\sum_{s=1}^{n} T_\nu^{\mathrm{f}}(\lambda_{\nu,s}^{\mathrm{f}}) f(\lambda_{\nu,s}^{\mathrm{f}}) - \int_0^b dx\, f(x)\right] = \frac{\pi}{2}\operatorname*{Res}_{z=0} f(z)\frac{\tilde{Y}_\nu(z)}{\tilde{J}_\nu(z)}$$
$$-\frac{1}{\pi}\int_0^\infty dx\left[e^{-\nu\pi i} f(xe^{\pi i/2})\frac{K_\nu^{(+)}(x)}{I_\nu^{(+)}(x)} + e^{\nu\pi i} f(xe^{-\pi i/2})\frac{K_\nu^{(-)}(x)}{I_\nu^{(-)}(x)}\right], \tag{13.17}$$

where on the left $\lambda_{\nu,n}^{\mathrm{f}} < b < \lambda_{\nu,n+1}^{\mathrm{f}}$. In formula (13.17) we use the notations

$$F^{(\pm)}(z) = zF'(z) + (\mu_a - \sqrt{z^2 e^{\pm\pi i} + \mu_a^2} - n_\sigma \nu)F(z) \tag{13.18}$$

for a given function $F(z)$.

We apply formula (13.17) to the series over s in formula (13.10) for the VEVs of the energy density and vacuum stresses. As it can be seen from expressions (13.11)–(13.13), the corresponding functions $f(z)$ satisfy the relation

$$e^{-\nu\pi i} f(xe^{\pi i/2}) = -e^{\nu\pi i} f(xe^{-\pi i/2}), \quad \text{for} \quad 0 \leqslant x < \mu_a\, . \tag{13.19}$$

By taking into account that for these values x one has $F^{(+)}(x) = F^{(-)}(x)$, we conclude that the part of the integral on the right of Eq. (13.17) over the interval $(0, \mu_a)$ vanishes.

As a result, after the application of summation formula (13.17), the components of the vacuum EMT can be presented in the form

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle_{\mathrm{m}} + \langle T_i^k\rangle_{\mathrm{b}}, \tag{13.20}$$

where $\langle T_i^k\rangle_{\mathrm{m}}$ does not depend on the radius of the sphere a and is the contribution due to unbounded global monopole spacetime. The corresponding quantities for the massless case are investigated in [115]. The second term on the right of formula (13.20) is induced by the presence of the spherical shell and can be presented in the form (no summation over i) [54]

$$\langle T_i^i\rangle_{\mathrm{b}} = \frac{1}{\pi^2\alpha^2 a^3 r}\sum_{l=1}^\infty l\int_{\mu_a}^\infty \frac{x^3 dx}{\sqrt{x^2-\mu_a^2}}\frac{W\left[I_\nu(x), K_\nu(x)\right]}{W\left[I_\nu(x), I_\nu(x)\right]} F_\nu^{(i)}\left[x, I_\nu(xr/a)\right], \tag{13.21}$$

with

$$\begin{aligned} F_\nu^{(0)}\left[x, I_\nu(y)\right] &= \left(1 - \frac{\mu_a^2}{x^2}\right)\left\{I_{\nu-1}^2(y) - I_\nu^2(y) - \mu_a \frac{I_{\nu-1}^2(y) + I_\nu^2(y)}{W\left[I_\nu(x), K_\nu(x)\right]}\right\}, &(13.22)\\ F_\nu^{(1)}\left[x, I_\nu(y)\right] &= I_{\nu-1}^2(y) - I_\nu^2(y) - \frac{2\nu-1}{y} I_\nu(y) I_{\nu-1}(y). &(13.23)\end{aligned}$$

Here and below $l = j + 1/2$, $\nu \equiv \nu_1 = l/\alpha + 1/2$, and for given functions $f(x)$ and $g(x)$ we use the notation

$$\begin{aligned} W\left[f(x), g(x)\right] &= \left[xf'(x) + (\mu_a + \nu)f(x)\right] \\ &\quad \times \left[xg'(x) + (\mu_a + \nu)g(x)\right] + (x^2 - \mu_a^2)f(x)g(x). \end{aligned} \tag{13.24}$$

It can be easily checked that for a massless spinor field the boundary-induced part of the vacuum EMT is traceless and the trace anomalies are contained only in the purely global monopole part without boundaries. These expressions diverge in a non-integrable manner as the boundary is approached. The energy density and azimuthal pressure vary, to leading order, as the inverse cube of the distance from the sphere. At the sphere center the boundary parts vanish for the global monopole spacetime ($\alpha < 1$) and are finite for the Minkowski spacetime ($\alpha = 1$). In the limit of strong gravitational field, corresponding to small values of the parameter α, describing the solid angle deficit, the boundary-induced part of the vacuum EMT is suppressed by the factor $\exp[-(2/\alpha)|\ln(r/a)|]$ and the corresponding vacuum stresses are strongly anisotropic: $\langle T_1^1\rangle_b \sim \alpha \langle T_2^2\rangle_b$. Having the components of the energy-momentum tensor we can find the corresponding fermionic condensate $\langle 0|\bar{\psi}\psi|0\rangle$ making use of formula $T_\mu^\mu = M\bar{\psi}\psi$ for the trace of the EMT. Note that the formula for the VEV of the EMT in the exterior region ($r > a$) is obtained from (13.21) by the replacement $I_\nu \to K_\nu$ in the denominator and in the argument of the function $F_\nu^{(i)}$ [54].

Fermionic vacuum densities induced by a spherical shell in the Minkowski bulk are obtained from the results of this section as a special case with $\alpha = 1$. The previous investigations on the spinor Casimir effect for a spherical boundary (see, for instance, [3, 4, 6, 7, 116] and references therein) consider mainly global quantities, such as total vacuum energy. For the case of a massless spinor the density of the fermionic condensate $\langle 0|\bar{\psi}\psi|0\rangle$ is investigated in [117] (see also [7]).

13.2 Vacuum expectation values of the energy-momentum tensor between two spherical shells

In this subsection we consider the region between two spherical shells concentric with the global monopole on which the fermionic field obeys bag boundary conditions:

$$\left(1 + i\gamma^l n_l^{(w)}\right) \psi\big|_{r=w} = 0 \ , \ w = a, b, \tag{13.25}$$

where a and b are the radii for the spheres, $a < b$, $n_l^{(w)} = n^{(w)}\delta_l^1$ is the outward-pointing normal to the boundaries. Here and below we use the notations $n^{(a)} = -1$, $n^{(b)} = 1$. The corresponding eigenfunctions have the form (13.4). Note that in terms of the function $Z_{\nu_\sigma}(kr)$ the boundary conditions (13.25) take the form

$$Z_{\nu_\sigma}(kw) = -Z_{\nu_\sigma+n_\sigma}(kw)\frac{n^{(w)}n_\sigma k}{\sqrt{k^2+M^2}-M}, \tag{13.26}$$

with $w = a, b$. In the region between two spherical shells, the function $Z_{\nu_\sigma}(kr)$ is a linear combination of the Bessel and Neuamnn functions. The coefficient in this linear combination is determined from the boundary condition (13.26) on the sphere $r = a$ and one obtains

$$Z_{\nu_\sigma}(kr) = g_{\nu_\sigma}^{(a)}(ka, kr) \equiv J_{\nu_\sigma}(kr)\tilde{Y}_{\nu_\sigma}^{(a)}(ka) - Y_{\nu_\sigma}(kr)\tilde{J}_{\nu_\sigma}^{(a)}(ka), \tag{13.27}$$

where for a given function $F(z)$ we use the notation

$$\tilde{F}^{(w)}(z) \equiv zF'(z) + [n^{(w)}(\mu_w - \sqrt{z^2+\mu_w^2}) - n_\sigma\nu_\sigma]F(z) \ , \tag{13.28}$$

with $w = a, b$, and $\mu_w = Mw$. Now, from the boundary condition on the outer sphere one finds that the eigenvalues for k are solutions to the equation

$$C_{\nu_\sigma}^{\mathrm{f}}(b/a, ka) \equiv \tilde{J}_{\nu_\sigma}^{(a)}(ka)\tilde{Y}_{\nu_\sigma}^{(b)}(kb) - \tilde{Y}_{\nu_\sigma}^{(a)}(ka)\tilde{J}_{\nu_\sigma}^{(b)}(kb) = 0. \tag{13.29}$$

Below we denote by $\gamma^{\rm f}_{\nu_\sigma,s}=ka$, $s=1,2,\ldots$, the positive roots to equation (13.29), arranged in ascending order, $\gamma^{\rm f}_{\nu_\sigma,s}<\gamma^{\rm f}_{\nu_\sigma,s+1}$. Substituting the eigenfunctions into the normalization integral and using the standard integrals for cylinder functions (see, for instance, [66]), for the normalization coefficient of the negative frequency modes one finds

$$A_\sigma^2=\frac{\pi^2k(\sqrt{k^2+M^2}-M)}{8\alpha^2a\sqrt{k^2+M^2}}T_\nu^{{\rm f}ab}(\eta,ka),\ ka=\gamma^{\rm f}_{\nu_\sigma,s},\tag{13.30}$$

where we have introduced the notation

$$T_\nu^{{\rm f}ab}(\eta,z)=z\left[\frac{\tilde{J}_\nu^{(a)2}(z)}{\tilde{J}_\nu^{(b)2}(\eta z)}D_\nu^{(b)}-D_\nu^{(a)}\right]^{-1},\tag{13.31}$$

with

$$D_\nu^{(w)}=w^2\left[k^2+(M-E)(M-n_\sigma\nu n^{(w)}/w)+\frac{n^{(w)}k^2}{2wE}\right],\tag{13.32}$$

and $ka=z$. Substituting eigenfunctions (13.4) with (13.27) into Eq. (13.3), the summation over the quantum number m can be done by using the standard summation formula for the spherical harmonics. For the EMT components one finds (no summation over i)

$$\langle 0|T_i^k|0\rangle=-\pi\delta_i^k\sum_{j=1/2}^{\infty}\frac{2j+1}{32\alpha^2a^3r}\sum_{\sigma=0,1}\sum_{s=1}^{\infty}T_{\nu_\sigma}^{{\rm f}ab}(\eta,z)f^{(i)}_{\sigma\nu_\sigma}[z,g^{(a)}_{\nu_\sigma}(z,zr/a)]_{z=\gamma^{\rm f}_{\nu_\sigma,s}},\tag{13.33}$$

where the functions $f^{(i)}_{\sigma\nu}[z,g_\nu(y)]$ are defined by formulae (13.11)-(13.13) with

$$g_{\nu+n_\sigma}(y)=g^{(a)}_{\nu+n_\sigma}(z,y)\equiv J_{\nu+n_\sigma}(y)\tilde{Y}_\nu^{(a)}(z)-Y_{\nu+n_\sigma}(y)\tilde{J}_\nu^{(a)}(z).\tag{13.34}$$

The VEV given by formula (13.33) is divergent and needs some regularization procedure. To make it finite we can introduce a cutoff function $\Phi_\lambda(z)$, $z=\gamma_{\nu_\sigma,s}$, with the cutoff parameter λ, which decreases sufficiently fast with increasing z and satisfies the condition $\Phi_\lambda\to 1$, $\lambda\to 0$.

To evaluate the VEV of the EMT we need to sum series over the zeros of the function $C_\nu^{\rm f}(\eta,z)$. A summation formula for this type of series can be obtained by making use of the GAPF. In the GAPF as functions $g(z)$ and $f(z)$ we choose

$$g(z)=\frac{1}{2i}\left[\frac{\tilde{H}_\nu^{(1b)}(\eta z)}{\tilde{H}_\nu^{(1a)}(z)}+\frac{\tilde{H}_\nu^{(2b)}(\eta z)}{\tilde{H}_\nu^{(2a)}(z)}\right]\frac{h(z)}{C_\nu^{\rm f}(\eta,z)},\quad f(z)=\frac{h(z)}{\tilde{H}_\nu^{(1a)}(z)\tilde{H}_\nu^{(2a)}(z)},\tag{13.35}$$

with the sum and difference

$$g(z)-(-1)^nf(z)=-i\frac{\tilde{H}_\nu^{(na)}(\eta z)}{\tilde{H}_\nu^{(na)}(z)}\frac{h(z)}{C_\nu^{\rm f}(\eta,z)},\quad n=1,2.\tag{13.36}$$

The conditions for the GAPF written in terms of the function $h(z)$ are as follows

$$|h(z)|<\varepsilon_1(x)e^{c|y|},\quad |z|\to\infty,\quad z=x+iy,\tag{13.37}$$

where $c<2(\eta-1)$ and $\varepsilon_1(x)/x\to 0$ for $x\to+\infty$. To find the residues of the function $g(z)$ at the poles $z=\gamma^{\rm f}_{\nu,s}$ we need the derivative

$$\frac{\partial}{\partial z}C_\nu^{\rm f}(\eta,z)=\frac{4}{\pi T_\nu^{{\rm f}ab}(\eta,z)}\frac{\tilde{J}_\nu^{(b)}(\eta z)}{\tilde{J}_\nu^{(a)}(z)},\quad z=\gamma^{\rm f}_{\nu,s}.\tag{13.38}$$

By using this relation it can be seen that

$$\operatorname*{Res}_{z=\gamma_{\nu,s}^{\mathrm{f}}} g(z) = \frac{\pi}{4i} T_{\nu}^{\mathrm{f}ab}(\eta, \gamma_{\nu,s}^{\mathrm{f}}) h(\gamma_{\nu,s}^{\mathrm{f}}). \tag{13.39}$$

Let $h(z)$ be an analytic function for $\operatorname{Re} z \geqslant 0$ except possible branch points on the imaginary axis. By using the GAPF, it can be seen that if this function satisfies condition (13.37),

$$h(ze^{\pi i}) = -h(z) + o(z^{-1}), \quad z \to 0, \tag{13.40}$$

and the integral

$$\int_0^L \frac{h(x)dx}{\tilde{J}_{\nu}^{(a)2}(x) + \tilde{Y}_{\nu}^{(a)2}(x)} \tag{13.41}$$

exists, then [55]

$$\begin{aligned}\lim_{L\to+\infty} &\left[\frac{\pi^2}{4} \sum_{s=1}^{m} h(\gamma_{\nu,s}^{\mathrm{f}}) T_{\nu}^{\mathrm{f}ab}(\eta, \gamma_{\nu,s}^{\mathrm{f}}) - \int_0^L \frac{h(x)dx}{\tilde{J}_{\nu}^{(a)2}(x) + \tilde{Y}_{\nu}^{(a)2}(x)} \right] = \\ &= -\frac{\pi}{2} \operatorname*{Res}_{z=0} \left[\frac{h(z)\tilde{H}_{\nu}^{(1b)}(\eta z)}{C_{\nu}^{\mathrm{f}}(\eta, z)\tilde{H}_{\nu}^{(1a)}(z)} \right] - \frac{\pi}{4} \sum_{\beta=\pm} \int_0^{\infty} dx\, \Omega_{a\nu}^{(\beta)}(x, \eta x) h(xe^{\beta\pi i/2}),\end{aligned} \tag{13.42}$$

where $\gamma_{\nu,m}^{\mathrm{f}} < L < \gamma_{\nu,m+1}^{\mathrm{f}}$. Here the function $\Omega_{a\nu}^{(\beta)}(x, \eta x)$ is defined as

$$\Omega_{a\nu}^{(\beta)}(x, \eta x) = \frac{K_{\nu}^{(b\beta)}(\eta x)/K_{\nu}^{(a\beta)}(x)}{K_{\nu}^{(a\beta)}(x) I_{\nu}^{(b\beta)}(\eta x) - I_{\nu}^{(a\beta)}(x) K_{\nu}^{(b\beta)}(\eta x)}, \tag{13.43}$$

and for a given function $F(z)$ we use the notation

$$F^{(w\pm)}(z) = zF'(z) + [n^{(w)}(\mu_w - \sqrt{z^2 e^{\pm\pi i} + \mu_w^2}) - n_{\sigma}\nu] F(z) \; . \tag{13.44}$$

Now we apply to the sum over s in (13.33) the summation formula (13.42). As a function $h(z)$ in this formula we take $h(z) = f_{\sigma\nu_{\sigma}}^{(q)}[z, g_{\nu_{\sigma}}^{(a)}(z, zr/a)]\Phi_{\lambda}(z)$. The function $f(z) = f_{\sigma\nu_{\sigma}}^{(q)}[z, g_{\nu_{\sigma}}^{(a)}(z, zr/a)]$ satisfies the relation

$$f(xe^{\pi i/2}) = -f(xe^{-\pi i/2}) \; , \quad \text{for} \quad 0 \leqslant x \leqslant \mu_a \; . \tag{13.45}$$

By taking into account that for these values x one has $F^{(w+)}(wx/a) = F^{(w-)}(wx/a)$, we conclude that in this case the part of the integral on the right of Eq. (13.42) over the interval $(0, \mu_a)$ vanishes after removing the cutoff. As a result the components of the vacuum EMT can be presented in the form [55]

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle_{1a} + \langle T_i^k\rangle_{ab}, \tag{13.46}$$

with separate parts

$$\langle T_i^k\rangle_{1a} = \frac{-\delta_i^k}{8\pi\alpha^2 a^3 r} \sum_{j=1/2}^{\infty} (2j+1) \sum_{\sigma=0,1} \int_0^{\infty} dx \frac{f_{\sigma\nu_{\sigma}}^{(i)}[x, g_{\nu_{\sigma}}^{(a)}(x, xr/a)]}{\tilde{J}_{\nu_{\sigma}}^{(a)2}(x) + \tilde{Y}_{\nu_{\sigma}}^{(a)2}(x)}, \tag{13.47}$$

and

$$\langle T_i^k\rangle_{ab} = \frac{-\delta_i^k}{2\pi^2\alpha^2 r} \sum_{l=1}^{\infty} l \sum_{\beta=\pm} \int_M^{\infty} dx \Omega_{a\nu}^{(\beta)}(ax, bx) F_{1\nu}^{(i\beta)}[ax, G_{\nu}^{(a\beta)}(ax, rx)], \tag{13.48}$$

where $\nu \equiv \nu_1 = l/\alpha + 1/2$, and the notation (13.44) is specified to

$$F^{(w\pm)}(z) = zF'(z) + (\nu + n^{(w)}\mu_w \mp in^{(w)}\sqrt{z^2-\mu_w^2})F(z) \ . \tag{13.49}$$

In formula (13.48) we have introduced the notations

$$\begin{aligned} F_{1\nu}^{(0\beta)}[x, G_\nu^{(a\beta)}(ax,y)] &= x\Big[(\sqrt{x^2-M^2}+\beta iM)G_\nu^{(a\beta)2}(ax,y) \\ &\quad -(\sqrt{x^2-M^2}-\beta iM)G_{\nu-1}^{(a\beta)2}(ax,y)\Big] \end{aligned} \tag{13.50}$$

$$\begin{aligned} F_{1\nu}^{(1\beta)}[x, G_\nu^{(a\beta)}(ax,y)] &= \frac{x^3}{\sqrt{x^2-M^2}}\Big[G_\nu^{(a\beta)2}(ax,y) - G_{\nu-1}^{(a\beta)2}(ax,y) \\ &\quad +\frac{2\nu-1}{y}G_\nu^{(a\beta)}(ax,y)G_{\nu-1}^{(a\beta)}(ax,y)\Big] \end{aligned} \tag{13.51}$$

$$F_{\sigma\nu}^{(2\beta)}[x, G_\nu^{(a\beta)}(ax,y)] = -\frac{(2\nu-1)x^3}{2y\sqrt{x^2-M^2}}G_\nu^{(a\beta)}(ax,y)G_{\nu-1}^{(a\beta)}(ax,y) \ , \tag{13.52}$$

where

$$G_\nu^{(w\pm)}(x,y) = I_\nu(y)K_\nu^{(w\pm)}(x) - K_\nu(y)I_\nu^{(w\pm)}(x), \ w=a,b, \tag{13.53}$$

$$G_{\nu-1}^{(w\pm)}(x,y) = I_{\nu-1}(y)K_\nu^{(w\pm)}(x) + K_{\nu-1}(y)I_\nu^{(w\pm)}(x). \tag{13.54}$$

In these formulae, for a given function $F(z)$ we use the notation $F^{(w\pm)}(z)$ defined by formula (13.44).

As it has been shown in Ref. [54], the term (13.47) presents the vacuum EMT in the case of a single spherical shell with radius a in the region $r > a$. After the subtraction of the part coming from the global monopole geometry without boundaries, this term is presented in the form

$$\langle T_i^k\rangle_{1a} = \langle T_i^k\rangle_{\rm m} + \langle T_i^k\rangle_a, \tag{13.55}$$

where the part

$$\langle T_i^k\rangle_a = \frac{-\delta_i^k}{2\pi^2\alpha^2 r}\sum_{l=1}^\infty l\sum_{\beta=\pm}\int_M^\infty dx\frac{I_\nu^{(a\beta)}(ax)}{K_\nu^{(a\beta)}(ax)}F_{1\nu}^{(i\beta)}\left[x, K_\nu(rx)\right] \ , \tag{13.56}$$

is induced by a single sphere with radius a in the region $r > a$. This quantity diverges on the boundary $r = a$ with the leading divergence $(r-a)^{-3}$ for the energy density and the azimuthal stress, and $(r-a)^{-2}$ for the radial stress.

Note that by using the identities

$$\begin{aligned} \frac{I_\nu^{(a\beta)}(ax)}{K_\nu^{(a\beta)}(ax)}F_{1\nu}^{(i\beta)}\left[x, K_\nu(rx)\right] &= \frac{K_\nu^{(b\beta)}(bx)}{I_\nu^{(b\beta)}(bx)}F_{1\nu}^{(i\beta)}\left[x, I_\nu(rx)\right] \\ &+\sum_{w=a,b} n^{(w)}\Omega_{w\nu}^{(\beta)}(ax,bx)F_{1\nu}^{(q\beta)}[x, G_\nu^{(w\beta)}(wx,rx)], \end{aligned} \tag{13.57}$$

with the notation

$$\Omega_{b\nu}^{(\beta)}(ax,bx) = \frac{I_\nu^{(a\beta)}(ax)/I_\nu^{(b\beta)}(bx)}{K_\nu^{(a\beta)}(ax)I_\nu^{(b\beta)}(bx) - I_\nu^{(a\beta)}(ax)K_\nu^{(b\beta)}(bx)}, \tag{13.58}$$

the vacuum EMT in the region between the spheres can also be presented in the form

$$\langle T_i^k\rangle = \langle T_i^k\rangle_m + \langle T_i^k\rangle_b + \langle T_i^k\rangle_{ba}, \tag{13.59}$$

where

$$\langle T_i^k \rangle_{ba} = \frac{-\delta_i^k}{2\pi^2\alpha^2 r} \sum_{l=1}^{\infty} l \sum_{\beta=\pm} \int_M^{\infty} dx\, \Omega_{b\nu}^{(\beta)}(ax,bx) F_{1\nu}^{(i\beta)}[x, G_\nu^{(b\beta)}(bx,rx)], \tag{13.60}$$

and the quantities $\langle T_i^k \rangle_b$ are the vacuum densities induced by a single shell with radius b in the region $r < b$. The formula for the latter is obtained from (13.56) by the replacements $a \to b$, $I \rightleftarrows K$ and coincides with the result derived in the previous subsection. The surface divergences in vacuum expectation values of the EMT are the same as those for a single sphere when the second sphere is absent. In particular, the term $\langle T_i^k \rangle_{ab}$ ($\langle T_i^k \rangle_{ba}$) is finite on the outer (inner) sphere. In the formulae above taking $\alpha = 1$ we obtain the corresponding quantities for a spinor field in the Minkowski bulk. In this case $\nu = l + 1/2$ and the Bessel modified functions are expressed in terms of elementary functions.

As in the case of a scalar field, the vacuum forces acting on the spheres can be presented in the form of the sum of self-action and interaction parts. The latter is determined by formulae (13.48) and (13.60) with $i = k = 1$ substituting $r = a$ and $r = b$ respectively. By making use of the properties of the modified Bessel functions, the interaction force on the sphere $r = w$ is presented in the form

$$p_{(\mathrm{int})}^{(w)} = \frac{n^{(w)}}{\pi^2\alpha^2 w^2} \frac{\partial}{\partial w} \sum_{l=1}^{\infty} l \int_M^{\infty} dx \frac{x}{\sqrt{x^2 - M^2}} \ln \left| 1 - \frac{I_\nu^{(a+)}(ax) K_\nu^{(b+)}(bx)}{K_\nu^{(a+)}(ax) I_\nu^{(b+)}(bx)} \right|, \tag{13.61}$$

where $w = a, b$, and $n^{(w)}$ is defined after formula (13.25). As it has been shown in [55], these forces can also be obtained from the Casimir energy by standard differentiation with respect to the sphere radii.

14 Electromagnetic Casimir densities for conducting spherical boundaries

14.1 Energy-momentum tensor inside a spherical shell

In this section we consider the application of the GAPF to the investigation of the VEV for the EMT of the electromagnetic field induced by perfectly conducting spherical boundaries in the Minkowski spacetime. For the region inside a perfectly conducting sphere with radius a, in the Coulomb gauge the corresponding eigenfunctions for the vector potential are presented in the form

$$\mathbf{A}_\sigma = \beta_\sigma e^{-i\omega t} \left\{ \begin{array}{ll} \omega j_l(\omega r) \mathbf{X}_{lm}(\theta,\varphi), & \lambda = 0 \\ \nabla \times [j_l(\omega r) \mathbf{X}_{lm}(\theta,\varphi)], & \lambda = 1 \end{array} \right., \quad \sigma = (\omega l m \lambda), \tag{14.1}$$

where $\lambda = 0$ and 1 correspond to the spherical waves of magnetic and electric type (TE and TM modes respectively). They describe photon with definite values of the total angular momentum l, its projection m, energy ω and parity $(-1)^{l+\lambda+1}$. Here the spherical vector harmonics have the form

$$\mathbf{X}_{lm}(\theta,\varphi) = -i \frac{\mathbf{r} \times \nabla}{\sqrt{l(l+1)}} Y_{lm}(\theta,\varphi), \quad l \neq 0, \tag{14.2}$$

with $Y_{lm}(\theta,\varphi)$ being the spherical functions, and $j_l(x) = \sqrt{\pi/2x} J_{l+1/2}(x)$ is the spherical Bessel function. The coefficients β_σ are determined by the normalization condition

$$\int dV\, \mathbf{A}_\sigma \cdot \mathbf{A}_{\sigma'}^* = \frac{2\pi}{\omega} \delta_{\sigma\sigma'}, \tag{14.3}$$

where the integration goes over the region inside the sphere. Using the standard formulae for the spherical vector harmonics and spherical Bessel functions (see, for example, [118]), for the normalization coefficient one finds

$$\beta_\sigma^2 = 8T_\nu(\omega a)/\omega a, \quad \nu = l + 1/2, \tag{14.4a}$$

where $T_\nu(z)$ is defined in (3.8).

Inside the perfectly conducting sphere the photon energy levels are quantized by the standard boundary conditions:

$$\mathbf{n} \times \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0, \quad r = a, \tag{14.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields and $\mathbf{n}$ is the normal to the boundary. They lead to the following eigenvalue equations with respect to ω:

$$\partial_r^\lambda \left[r j_l(\omega r) \right]_{r=a} = 0, \quad \lambda = 0, 1. \tag{14.6}$$

It is well known that these equations have an infinite number of real simple roots.

By substituting the eigenfunctions into the mode-sum formula

$$\langle 0 | T_{ik} | 0 \rangle = \sum_\sigma T_{ik} \left\{ \mathbf{A}_\sigma, \mathbf{A}_\sigma^* \right\}, \tag{14.7}$$

with the standard expression for the electromagnetic EMT and after the summation over m by using the standard formulae for the vector spherical harmonics (see, for example, [118]), one obtains

$$\langle 0 | T_i^k | 0 \rangle = \mathrm{diag}\,(\varepsilon,\, -p,\, -p_\perp,\, -p_\perp) \tag{14.8}$$

(index values 1,2,3 correspond to the spherical coordinates r, θ, φ with the origin at the sphere center). Here the energy density, ε, effective pressures in transverse, $p_\perp$, and radial, p, directions are determined by the relations

$$q(a, r) = \sum_{\omega l \lambda} \frac{2l+1}{4\pi^2 a} \omega^3 T_\nu(\omega a) D_l^{(q)}[j_l(\omega r)], \quad q = \varepsilon,\, p,\, p_\perp, \tag{14.9}$$

where the following notations are introduced

$$D_l^{(q)}[f(y)] = \begin{cases} \left[y f(y) \right]'^2 / y^2 + [1 + l(l+1)/y^2] f^2(y), & q = \varepsilon, \\ l(l+1) f^2(y)/y^2, & q = p_\perp, \end{cases} \tag{14.10}$$

and $p = \varepsilon - 2p_\perp$. In the sum (14.9), ω takes a discrete set of values determined by equations (14.6).

The VEVs (14.9) are infinite. The renormalization of $\langle 0 | T_{ik} | 0 \rangle$ in flat spacetime is affected by subtracting from this quantity its singular part $\langle 0_{\mathrm{M}} | T_{ik} | 0_{\mathrm{M}} \rangle$, which is precisely the value it would have if the boundary is absent. Here $|0_{\mathrm{M}}\rangle$ is the amplitude for the Minkowski vacuum state. To evaluate the finite difference between these two infinities we will introduce a cutoff function $\psi_\mu(\omega)$, which makes the sums finite and satisfies the condition $\psi_\mu(\omega) \to 1$, $\mu \to 0$. After the subtraction we will allow $\mu \to 0$:

$$\langle T_{ik} \rangle_{\mathrm{ren}} = \lim_{\mu \to 0} \left[\langle 0 | T_{ik} | 0 \rangle - \langle 0_{\mathrm{M}} | T_{ik} | 0_{\mathrm{M}} \rangle \right]. \tag{14.11}$$

Hence, we consider the following finite quantities

$$q(\mu, a, r) = \sum_{l=1}^{\infty} \frac{2l+1}{4\pi^2 a^4} \sum_{\lambda=0,1} \sum_{k=1}^{\infty} j_{\nu,k}^{(\lambda)3} T_\nu(j_{\nu,k}^{(\lambda)}) \psi_\mu(j_{\nu,k}^{(\lambda)}/a) D_l^{(q)}[j_l(j_{\nu,k}^{(\lambda)} x)], \tag{14.12}$$

where $x = r/a$, and $\omega = j_{\nu,k}^{(\lambda)}/a$ are solutions to the eigenvalue equations (14.6) for $\lambda = 0, 1$, respectively. The summation over k in (14.12) can be done by using formula (3.22) and taking $A = 1, B = 0$ for TE modes ($\lambda = 0$) and $A = 1, B = 2$ for TM modes ($\lambda = 1$) (recall that in (3.18) $\lambda_{\nu,k}$ are zeros of $\bar{J}_\nu(z)$ with barred quantities defined as (3.1)). Let us substitute in formula (3.22)

$$f(z) = z^3\psi_\mu(z/a)D_l^{(q)}[j_l(zx)], \tag{14.13}$$

with $D_l^{(q)}[f(y)]$ defined from (14.10). We will assume the class of cutoff functions for which the function (14.13) satisfies the conditions for Theorem 2, uniformly with respect to μ (the corresponding restrictions for ψ_μ can be easily found from these conditions using the asymptotic formulae for the Bessel functions). Below for simplicity we will consider the functions without poles. In this case, function (14.13) is analytic on the right-half plane of the complex variable z. The discussion on the conditions to cutoff functions, under which the difference between divergent sum and integral exists and has a finite value independent any further details of cutoff function, can be found in [11]. For TE and TM modes by choosing the constants A and B as mentioned above one obtains

$$\begin{aligned} q &= \frac{1}{8\pi^2}\sum_{l=1}^{\infty}(2l+1)\left\{2\int_0^\infty d\omega\, \omega^3\psi_\mu(\omega)D_l^{(q)}[j_l(\omega r)]\right. \\ &\left. -\frac{1}{r^2}\int_0^\infty dz\, z\chi_\mu(z)\left[\frac{e_l(az)}{s_l(az)}+\frac{e_l'(az)}{s_l'(az)}\right]F_l^{(q)}[s_l(zr)]\right\}, \end{aligned} \tag{14.14}$$

where the functions $F_l^{(q)}[f(y)]$ are defined as

$$F_l^{(\varepsilon)}[f(y)] = f'^2(y) + [l(l+1)/y^2 - 1]f^2(y), \tag{14.15}$$

$$F_l^{(p_\perp)}[f(y)] = l(l+1)\frac{f^2(y)}{y^2}, \quad \chi_\mu(y) = [\psi_\mu(iy)+\psi_\mu(-iy)]/2, \tag{14.16}$$

and $F_l^{(p)}[f(y)] = F_l^{(\varepsilon)}[f(y)] - 2F_l^{(p_\perp)}[f(y)]$. In (14.14) we have introduced the Ricatti-Bessel functions of the imaginary argument,

$$s_l(z) = \sqrt{\frac{\pi z}{2}}I_\nu(z), \quad e_l(z) = \sqrt{\frac{2z}{\pi}}K_\nu(z),\ \nu = l+1/2. \tag{14.17}$$

As $\langle 0_{\rm M}|T_{ik}|0_{\rm M}\rangle = \lim_{a\to\infty}\langle 0|T_{ik}|0\rangle$, the first integral in (14.14) represents the vacuum EMT for empty Minkowski spacetime:

$$q_{\rm M} = \frac{1}{4\pi^2}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty d\omega\, \omega^3\psi_\mu(\omega)D_l^{(q)}[j_l(\omega r)]. \tag{14.18}$$

This expression can be further simplified. For example, in the case of the energy density one has

$$\begin{aligned} \varepsilon_{\rm M} &= \frac{1}{4\pi^2}\sum_{l=1}^{\infty}\int_0^\infty d\omega\, \omega^3\psi_\mu(\omega)\left[lj_{l+1}^2(\omega r)+(l+1)j_{l-1}^2(\omega r)+(2l+1)j_l^2(\omega r)\right] \\ &= \frac{1}{2\pi^2}\int_0^\infty d\omega\, \omega^3\psi_\mu(\omega)\sum_{l=0}^{\infty}(2l+1)j_l^2(\omega r) = \int_0^\infty d\omega\, \omega^3\psi_\mu(\omega). \end{aligned} \tag{14.19}$$

As we see, the use of the GAPF allows to extract from infinite quantities the divergent part without specifying the form of the cutoff function. Now the renormalization of the EMT is

equivalent to omitting the first summand in (14.14), which corresponds to the contribution of the spacetime without boundaries. For $r < a$ the second term on the right (14.14) is finite in the limit $\mu \to 0$ and for the renormalized components one obtains

$$q_{\rm ren}(a,r) = -\sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 r^2} \int_0^{\infty} dz\, z \left[\frac{e_l(az)}{s_l(az)} + \frac{e_l'(az)}{s_l'(az)} \right] F_l^{(q)}[s_l(zr)], \quad r < a. \tag{14.20}$$

From here it is obvious the independence of the renormalized quantities on the specific form of the cutoff, on the class of functions for which (14.13) satisfies conditions for (3.22). The derivation of the vacuum densities (14.20) given above uses the GAPF to summarize mode-sums and is based on [34, 35]. One can see that these formulae for the case of exponential cutoff function may also be obtained from the results of [105, 106], where the Green function method is used.

We obtained the renormalized values (14.20) by introducing a cutoff function and subsequently subtracting the contribution due to the unbounded space. The GAPF in the form (3.22) allows to obtain immediately this finite difference. It should be noted that by using the GAPF in the form (3.18) we can derive the expressions for the renormalized azimuthal pressure without introducing any special cutoff function. To see this note that for $x < 1$ the function (14.13) with $q = p_\perp$ and $\psi_\mu = 1$ satisfies conditions of Theorem 2. It follows from here that we can apply formula (3.18) directly to the corresponding sum over ω in (14.9) or over k in (14.12) without introducing a cutoff function. This immediately yields to formula (14.20) for $q = p_\perp$ with $\psi_\mu = 1$.

The VEV given by formula (14.20) diverges on the sphere surface, $r = a$, and this divergence is due to the contribution of large l (note that the integral over z converges at $r = a$). The corresponding asymptotic behavior can be found by using the uniform asymptotic expansions for the Bessel functions and the leading terms have the form

$$\varepsilon \approx 2p_\perp \approx \frac{-1}{30\pi^2 a (a-r)^3}, \quad p \approx \frac{-1}{60\pi^2 a^2 (a-r)^2}. \tag{14.21}$$

These surface divergences originate in the unphysical nature of perfect conductor boundary conditions and are well-known in quantum field theory with boundaries. As we have mentioned before, they are investigated in detail for various types of fields and general shape of the boundary. Eqs. (14.21) are particular cases of the asymptotic expansions for the VEV near the smooth boundary given in [82]. In reality the expectation values for the EMT components will attain a limiting value on the conductor surface, which will depend on the molecular details of the conductor. From the asymptotic expansions given above it follows that the main contributions to $q_{\rm ren}(r)$ are due to the frequencies $\omega < (a-r)^{-1}$. Hence we expect that formulae (14.20) are valid for real conductors up to distances r for which $(a-r)^{-1} \ll \omega_0$, with ω_0 being the characteristic frequency, such that for $\omega > \omega_0$ the conditions for perfect conductivity fail.

At the sphere center, in (14.20) $l = 1$ multipole contributes only and we obtain [35, 103]

$$\varepsilon(0) = 3p(0) = 3p_\perp(0) \approx -0.0381 a^{-4}. \tag{14.22}$$

At the center the equation of state for the electromagnetic vacuum is the same as that for black-body radiation. Note that the corresponding results obtained by using the uniform asymptotic expansions for the Bessel functions [105, 106] are in good agreement with (14.22).

The components of the renormalized EMT satisfy continuity equation $\nabla_k T_i^k = 0$, which for the spherical geometry takes the form

$$p'(r) + 2(p - p_\perp)/r = 0. \tag{14.23}$$

From here by using the zero trace condition the following integral relations may be obtained

$$p(r) = \frac{1}{r^3}\int_0^r dt\, t^2\varepsilon(t) = \frac{2}{r^2}\int_0^r dt\, tp_\perp(t), \tag{14.24}$$

where the integration constant is determined from relations (14.22) at the sphere center. It follows from the first relation that the total energy within a sphere with radius r is equal to

$$E(r) = 4\pi\int_0^r dt\, t^2\varepsilon(t) = 3V(r)p(r), \tag{14.25}$$

where $V(r)$ is the corresponding volume.

The distribution for the vacuum energy density and pressures inside the perfectly conducting sphere can be obtained from the results of the numerical calculations given in [105, 106]. In their calculations Brevik and Kolbenstvedt used the uniform asymptotic expansions of the Ricatti-Bessel functions for large values of the order. In [34, 35] (see also [28]) the corresponding quantities are calculated on the base of the exact relations for these functions and the accuracy of the numerical results in [105, 106] is estimated ($\approx 5\%$). The simple approximation formulae are presented with the same accuracy as asymptotic expressions. Note that inside the sphere all quantities ε, p, $p_\perp$ are negative.

By the method similar to that used in this subsection, the VEVs for gluon fields can be evaluated in the bag model for hadrons [119]. In this model the vacuum outside the spherical bag is a perfect color magnetic conductor. The color electric and magnetic fields are confined inside the bag and satisfy the boundary conditions $\mathbf{n}\cdot\mathbf{E} = 0$, $\mathbf{n}\times\mathbf{B} = 0$ on its boundary (for the role of the Casimir effect in the bag model see [3, 4, 7]).

14.2 Electromagnetic vacuum in the region between two spherical shells

Now let us consider the electromagnetic vacuum in the region between two concentric conducting spherical shells with radii a and b, $a < b$. In the Coulomb gauge the complete set of solutions to the Maxwell equations can be written in the form similar to (14.1):

$$\mathbf{A}_\sigma = \beta_\sigma \frac{e^{-i\omega t}}{\sqrt{4\pi}}\left\{\begin{array}{ll} \omega g_{0l}(\omega a, \omega r)\mathbf{X}_{lm}, & \lambda = 0 \\ \nabla\times[g_{1l}(\omega a, \omega r)\mathbf{X}_{lm}], & \lambda = 1 \end{array}\right., \tag{14.26}$$

where as above the values $\lambda = 0$ and $\lambda = 1$ correspond to the waves of magnetic (TE modes) and electric (TM modes) type,

$$g_{\lambda l}(x, y) = \left\{\begin{array}{ll} j_l(y)n_l(x) - j_l(x)n_l(y), & \lambda = 0 \\ j_l(y)[xn_l(x)]' - [xj_l(x)]' n_l(y), & \lambda = 1 \end{array}\right., \tag{14.27}$$

with $n_l(x)$ being the spherical Neumann function. From the boundary conditions at surfaces $r = a$ and $r = b$ one finds that the possible energy levels of the photon are solutions to the following equations

$$\partial_r^\lambda \left[rg_{\lambda l}(\omega a, \omega r)\right]_{r=b} = 0, \quad \lambda = 0, 1. \tag{14.28}$$

All roots of these equations are real and simple [65].

The coefficient β_σ in (14.26) is determined from the normalization condition (14.3), where now the integration goes over the region between spherical shells, $a \leqslant r \leqslant b$. By using the standard relations for the spherical Bessel functions it can be presented in the form

$$\beta_\sigma^2 = \omega a\left[\frac{aj_l^2(\omega a)}{bj_l^2(\omega b)} - 1\right]^{-1}, \quad \lambda = 0, \tag{14.29}$$

$$\beta_\sigma^2 = \frac{1}{\omega a}\left\{\frac{b\left[\omega a j_l(\omega a)\right]'^2}{a\left[\omega b j_l(\omega b)\right]'^2}\left[1 - \frac{l(l+1)}{\omega^2 b^2}\right] - 1 + \frac{l(l+1)}{\omega^2 a^2}\right\}^{-1}, \quad \lambda = 1. \tag{14.30}$$

From the mode-sum formula (14.7) with the functions (14.26) as a complete set of solutions one obtains the VEV in the form (14.8) with

$$q = \frac{1}{8\pi} \sum_{\omega l \lambda} (2l+1)\omega^4 \beta_\alpha^2 D_l^{(q)}[g_{\lambda l}(\omega a, \omega r)], \quad q = \varepsilon,\, p,\, p_\perp, \tag{14.31}$$

where the frequencies ω are solutions to equations (14.28) and the functions $D_l^{(q)}[f(y)]$ are defined by formulae (14.10) with $f(y) = g_{\lambda l}(\omega a, y)$. It is easy to see that the eigenvalue equations (14.28) can be written in terms of the function C_ν^{ab}, defined by (4.1), as

$$C_\nu^{ab}(\eta, \omega a) = 0, \quad \nu = l + 1/2,\, \eta = b/a,\, A = 1/(1+\lambda),\, B = \lambda,\, \lambda = 0, 1. \tag{14.32}$$

By this choice of constants A_j and B_j the normalization coefficients (14.29) and (14.30) are related to the function T_ν^{ab} from (4.7) by the formula

$$\beta_\sigma^2 = T_\nu^{ab}(\eta, \omega a). \tag{14.33}$$

This allows us to use the formulae from Section 4 for the summation over the eigenmodes.

As above, to regularize infinite quantities (14.31) we introduce a cutoff function $\psi_\mu(\omega)$ and consider the finite quantities

$$q = \sum_{l=1}^{\infty} \frac{2l+1}{8\pi a^4} \sum_{\lambda=0}^{1} \sum_{k=1}^{\infty} z^4 T_\nu^{AB}(\eta, z)\psi_\mu(z/a) D_l^{(q)}[g_{\lambda l}(z, zx)]\big|_{z=\gamma_{\nu,k}^{(\lambda)}}, \tag{14.34}$$

where $x = r/a$, and $\omega a = \gamma_{\nu,k}^{(\lambda)}$ are solutions to equations (14.28) or (14.32). In order to sum over k we will use formula (4.14) with

$$h(z) = z^4 \psi_\mu(z/a) D_l^{(q)}[g_{\lambda l}(z, zx)], \tag{14.35}$$

assuming a class of cutoff functions for which (14.35) satisfies conditions (4.5) and (4.12) uniformly with respect to μ. The corresponding restrictions on ψ_μ can be obtained using the asymptotic formulae for Bessel functions. From (14.34), by applying to the sum over k formula (4.14), for the EMT components one obtains

$$\begin{aligned} q &= \sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 a^4} \sum_{\lambda=0,1} \left\{ \int_0^\infty dz\, z^3 \psi_\mu(z/a) \frac{D_l^{(q)}[g_{\lambda l}(z, zx)]}{\Omega_{1\lambda l}(z)} \right. \\ &\quad \left. + \frac{1}{x^2} \int_0^\infty dz\, \frac{e_l^{(\lambda)}(\eta z)}{e_l^{(\lambda)}(z)} \frac{z \chi_\mu(z/a) F_l^{(q)}[G_{\lambda l}(z, zx)]}{\left[\partial_y^\lambda G_{\lambda l}(z, y)\right]_{y=z\eta}} \right\}, \end{aligned} \tag{14.36}$$

where we use the notations

$$\begin{aligned} e_l^{(\lambda)}(y) &\equiv \partial_y^\lambda e_l(y), \quad s_l^{(\lambda)}(y) \equiv \partial_y^\lambda s_l(y) & (14.37) \\ G_{\lambda l}(x, y) &= e_l^{(\lambda)}(x) s_l(y) - e_l(y) s_l^{(\lambda)}(x), & (14.38) \end{aligned}$$

for the Riccati-Bessel functions,

$$\Omega_{1\lambda l}(z) = \begin{cases} j_l^2(z) + n_l^2(z), & \lambda = 0 \\ [zj_l(z)]'^2 + [zn_l(z)]'^2, & \lambda = 1 \end{cases}, \tag{14.39}$$

and the functions $F_l^{(q)}[f(y)]$ with $f(y) = G_{\lambda l}(z, y)$ are defined by relations (14.15), (14.16).

In (14.36), taking the limit $b \to \infty$ for fixed a and r, we obtain the VEV for the EMT components outside a single conducting spherical shell with radius a. In this limit the second integral on the right of formula (14.36) tends to zero, whereas the first one does not depend on b. Hence, one obtains

$$q_{b\to\infty} = \sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 a^4} \sum_{\lambda=0,1} \int_0^{\infty} dz\, z^3 \psi_\mu(z/a) \frac{D_l^{(q)}[g_{\lambda l}(z,zx)]}{\Omega_{1\lambda l}(z)}. \tag{14.40}$$

For the renormalization of expressions (14.40) we subtract the Minkowskian part, namely expression (14.18). It can be seen that

$$\frac{D_l^{(q)}[g_{\lambda l}(z,zx)]}{\Omega_{1\lambda l}(z)} - D_l^{(q)}[j_l(zx)] = -\frac{1}{2} \sum_{m=1,2} \frac{\partial_z^\lambda [z j_l(z)]}{\partial_z^\lambda [z h_l^{(m)}(z)]} D_l^{(q)}[h_l^{(m)}(zx)], \tag{14.41}$$

with $h_l^{(m)}(z)$, $m = 1,2$ being the spherical Hankel functions. Now, for the corresponding z-integral we rotate the integration contour in the complex z-plane by the angle $\pi/2$ for the term with $m = 1$ and by the angle $-\pi/2$ for the term with $m = 2$. By introducing the Ricatti-Bessel functions (14.17) and for points with $r > a$ removing the cutoff, for the renormalized components of the vacuum EMT outside the sphere we find

$$q_{\rm ren}(a,r) = -\sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 r^2} \int_0^{\infty} dz\, z \left[\frac{s_l(az)}{e_l(az)} + \frac{s_l'(az)}{e_l'(az)}\right] F_l^{(q)}[e_l(zr)], \quad r > a, \tag{14.42}$$

where the functions $F_l^{(q)}[f(y)]$ are defined by formulae (14.15), (14.16). The exterior mode-sum consideration given in this section follows [34, 36]. For the case of exponential cutoff function formula (14.42) can also be obtained from the results [105, 106], where the Green function formalism was used. Note that the expressions for the exterior components are obtained from the interior ones replacing $s_l \rightleftarrows e_l$.

Expressions (14.42) diverge on the sphere. The leading terms of these divergences may be found using the uniform asymptotic expansions for the modified Bessel functions for large values of the order and these terms are given by the same formulae as those for the interior region (see (14.21)). In particular, the cancellation of interior and exterior leading divergent terms occurs in calculating the total energy and force acting on the sphere. The same cancellations take place for the next subleading divergent terms as well. For distances far from the sphere one finds

$$p_\perp \approx \frac{a^3}{4\pi^2 r^7} \int_0^{\infty} dz\, z^2 e_1^2(z) = \frac{5a^3}{16\pi^2 r^7}, \quad \varepsilon \approx -4p \approx \frac{a^3}{2\pi^2 r^7}, \quad r \gg a. \tag{14.43}$$

The results of the numerical calculations for the vacuum EMT components outside the sphere are given in [35, 105, 106]. In [105, 106] calculations are carried out by using the uniform asymptotic expansions for the Riccati-Bessel functions. The accuracy of this approximation is estimated in [35], where exact relations are used in numerical calculations. Simple approximating formulae with the same accuracy as those for the asymptotic calculations are presented as well. In the exterior region the energy density and azimuthal pressure are positive, and radial pressure is negative.

Note that the continuity equation (20.33) may now be written in the following integral form

$$p(r) = \frac{1}{r^3} \int_{\infty}^{r} dt\, t^2 \varepsilon(t) = \frac{2}{r^2} \int_{\infty}^{r} dt\, t p_\perp(t). \tag{14.44}$$

From (14.24) and (14.44) it follows that

$$E(a)=\int dV\,\varepsilon(r)=4\pi a^3[p(a-)-p(a+)], \tag{14.45}$$

where $E(a)$ is the total vacuum energy for a spherical shell with radius a, $p(a\pm)=\lim_{r\to 0}p(a\pm r)$. By using the expressions for $p(r)$ given above, one can obtain the following formula for the total energy (the same result can also be obtained by integrating the energy density)

$$\begin{aligned} E(a) &= -\sum_{l=1}^{\infty}\frac{2l+1}{2\pi a}\int_0^{\infty}dz\chi_\mu(z/a)z\frac{[s_l(z)e_l(z)]'}{s_l'(z)e_l'(z)}\left[\frac{s_l'(z)e_l'(z)}{s_l(z)e_l(z)}+\frac{l(l+1)}{z^2}+1\right] \\ &= -\sum_{l=1}^{\infty}\frac{2l+1}{2\pi a}\int_0^{\infty}dz\chi_\mu(z/a)z\frac{d}{dz}\ln\left\{1-[s_l(z)e_l(z)]'^2\right\}, \end{aligned} \tag{14.46}$$

where we have restored the cutoff function. By taking the cutting function $\psi_\mu(\omega)=e^{-\mu\omega}$ one obtains the expression for the Casimir energy of the sphere derived in [99] by the Green function method. Note that in this method the factor $\psi_\mu(iz/a)=e^{-i\omega\mu}$ appears automatically as a result of the point splitting procedure. The evaluation of (14.46) leads to the result $E=0.092353/2a$ for the Casimir energy of a spherical conducting shell [81, 97, 98, 99, 100, 101]. This corresponds to the repulsive vacuum force on the sphere.

As we have seen, the VEV of the EMT in the region between two concentric perfectly conducting surfaces is given by (14.36). Using this formula the components of the vacuum EMT can be presented in the form

$$q(a,b,r)=q(a,r)+q^{(ab)}(r),\quad q=\varepsilon,p_\perp,p,\quad a<r<b, \tag{14.47}$$

where $q(a,r)$ are the corresponding quantities outside a single sphere of radius a given by (14.40), and [37]

$$q^{(ab)}(r)=\sum_{l=1}^{\infty}\frac{2l+1}{8\pi^2r^2}\sum_{\lambda=0,1}\int_0^{\infty}dz\,z\frac{F_l^{(q)}[G_{\lambda l}(az,rz)]e_l^{(\lambda)}(bz)/e_l^{(\lambda)}(az)}{e_l^{(\lambda)}(az)s_l^{(\lambda)}(bz)-e_l^{(\lambda)}(bz)s_l^{(\lambda)}(az)}. \tag{14.48}$$

In (14.47) the dependence on b is contained in the summand $q^{(ab)}(r)$ only. This quantity is finite for $a\leqslant r<b$ and the renormalization of $q(a,b,r)$ is equivalent to the renormalization of the first summand.

It can be seen that the quantities (14.47) may also be written in the form

$$q(a,b,r)=q(b,r)+q^{(ba)}(r),\quad q=\varepsilon,p_\perp,p,\quad a<r<b, \tag{14.49}$$

where

$$q^{(ba)}(r)=\sum_{l=1}^{\infty}\frac{2l+1}{8\pi^2r^2}\sum_{\lambda=0,1}\int_0^{\infty}dz\,z\frac{F_l^{(q)}[G_{\lambda l}(bz,rz)]s_l^{(\lambda)}(az)/s_l^{(\lambda)}(bz)}{e_l^{(\lambda)}(az)s_l^{(\lambda)}(bz)-e_l^{(\lambda)}(bz)s_l^{(\lambda)}(az)}. \tag{14.50}$$

In (14.49), $q^{(ba)}(r)\to 0$ when $a\to 0$ and $q(b,r)$ coincides with the corresponding quantities inside a single conducting shell with radius b. Note that in (14.50) the sum and integral are convergent for $a<r\leqslant b$.

As we have already mentioned the vacuum energy density and stresses inside and outside a single shell may be obtained from the expressions $q(a,b,r)$ in limiting cases $a\to 0$ or $b\to\infty$. It

can be seen that in the limit $a, b \to \infty$ with fixed $h = b - a$, we obtain the standard result for the Casimir parallel plate configuration with $\varepsilon = -\pi^2/720h^4$.

Let us present the quantities $q = \varepsilon, p, p_\perp$ in the form

$$q = q(a,r) + q(b,r) + \Delta q(a,b,r), \quad a < r < b, \tag{14.51}$$

where the interference term may be written in two ways

$$\Delta q(a,b,r) = q^{(ab)}(a,b,r) - q(b,r) = q^{(ba)}(a,b,r) - q(a,r). \tag{14.52}$$

This term is finite for all $a \leqslant r \leqslant b$. Near the surface $r = a$ it is convenient to use the first presentation in (14.52), as for $r \to a$ both summands in this formula are finite. For the same reason the second presentation is convenient for calculations near the surface $r = b$.

So far we have considered the electromagnetic vacuum in the region between two perfectly conducting spherical surfaces. Consider now a system consisting of two concentric thin spherical shells with radii a and b, $a < b$. In this case the VEV for the EMT components may be written in the form

$$q(a,b,r) = q(a,r)\theta(a-r) + q(b,r)\theta(r-b) + [q(a,r) + q^{(ab)}(r)]\theta(r-a)\theta(b-r), \tag{14.53}$$

where $\theta(x)$ is the unit step function. By using the continuity equation for the EMT, it is easy to see that the total Casimir energy for the system under consideration can be presented in the form

$$E(a,b) = E(a) + E(b) + 4\pi\left[b^3 p^{(ba)}(b) - a^3 p^{(ab)}(a)\right], \tag{14.54}$$

where $E(j)$ is the Casimir energy for a single sphere with radius $j = a, b$. As follows from (14.48) and (14.50), the additional vacuum pressures on the spheres are equal to [28, 37]

$$p^{(ab)}(a) = -\sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 a^4} \int_0^\infty dz\, z \sum_{\lambda=0,1} (-1)^\lambda \frac{\left[l(l+1)/z^2+1\right]^\lambda e_l^{(\lambda)}(bz/a)/e_l^{(\lambda)}(z)}{e_l^{(\lambda)}(z)s_l^{(\lambda)}(bz/a) - e_l^{(\lambda)}(bz/a)s_l^{(\lambda)}(z)}, \tag{14.55}$$

$$p^{(ba)}(b) = -\sum_{l=1}^{\infty} \frac{2l+1}{8\pi^2 b^4} \int_0^\infty dz\, z \sum_{\lambda=0,1} (-1)^\lambda \frac{\left[l(l+1)/z^2+1\right]^\lambda s_l^{(\lambda)}(az/b)/s_l^{(\lambda)}(z)}{e_l^{(\lambda)}(az/b)s_l^{(\lambda)}(z) - e_l^{(\lambda)}(z)s_l^{(\lambda)}(az/b)}. \tag{14.56}$$

In [120], the vacuum forces acting on boundaries in the geometry of concentric conducting spherical shells are investigated by making use of local Green function method. The results of the numerical evaluations of quantities $\Delta q(a,b,r)$, $q = \varepsilon, p, p_\perp$, as well as those for $p^{(ab)}(a)$, $p^{(ba)}(b)$ are presented in [28, 37]. Note that, as follows from the results of these calculations, the quantities (14.55) and (14.56) are always negative, and therefore the interaction forces between two spheres are always attractive (as in the parallel plate configuration). Note that the interaction forces can also be obtained from the corresponding part of the total Casimir energy in the region between the spheres by standard differentiation with respect to the sphere radii [120].

15 Vacuum polarization induced by a cylindrical boundary in the cosmic string spacetime

Cosmic strings generically arise within the framework of grand unified theories and could be produced in the early universe as a result of symmetry braking phase transitions [121, 122]. Although the recent observational data on the cosmic microwave background radiation have

ruled out cosmic strings as the primary source for primordial density perturbations, they are still candidates for the generation of a number interesting physical effects such as the generation of gravitational waves, high energy cosmic strings, and gamma ray bursts. In the simplest theoretical model describing the infinite straight cosmic string the spacetime is locally flat except on the string where it has a delta shaped curvature tensor. In quantum field theory the corresponding non-trivial topology leads to non-zero VEVs for physical observables. In this section we will consider the vacuum polarization induced by a cylindrical boundary coaxial with a cosmic string assuming that on the bounding surface the field obeys Robin boundary condition [43]. The cylindrical problem for the electromagnetic field in the Minkowski bulk with perfectly conducting conditions was first considered in [123] (see also [7, 124, 125]). While the earliest studies have focused on global quantities, such as the total energy and stress on a shell, the local characteristics of the corresponding electromagnetic vacuum are considered in [39] for the interior and exterior regions of a conducting cylindrical shell, and in [40] for the region between two coaxial shells (see also [28]). The vacuum forces acting on the boundaries in the geometry of two cylinders are also considered in [126]. The vacuum densities for a Robin cylindrical boundary in the Minkowski background are investigated in [41]. In [127] a cylindrical boundary with Dirichlet boundary condition is introduced in the bulk of the cosmic string as an intermediate stage for the calculation of the ground state energy of a massive scalar field in (2+1)-dimensions.

15.1 Wightman function

We consider a scalar field φ with curvature coupling parameter ζ propagating on the $(D+1)$-dimensional background spacetime with a conical-type singularity described by the line element

$$ds^2 = g_{ik}dx^i dx^k = dt^2 - dr^2 - r^2 d\phi^2 - \sum_{i=1}^{N} dz_i{}^2, \tag{15.1}$$

with the cylindrical coordinates $(x^1, x^2, \ldots, x^D) = (r, \phi, z_1, \ldots, z_N)$, where $N = D-2$, $0 \leqslant \phi \leqslant \phi_0$, and the spatial points $(r, \phi, z_1, \ldots, z_N)$ and $(r, \phi + \phi_0, z_1, \ldots, z_N)$ are to be identified. In the standard $D = 3$ cosmic string case the planar angle deficit is related to the mass per unit length of the string μ by $2\pi - \phi_0 = 8\pi G\mu$, where G is the Newton gravitational constant. We assume that the field obeys Robin boundary condition (9.11) on the cylindrical surface of radius a, coaxial with the string.

In the region inside the cylindrical surface the eigenfunctions satisfying the periodicity condition are specified by the set of quantum numbers $\sigma = (n, \gamma, \mathbf{k})$, $n = 0, \pm 1, \pm 2, \ldots$, $\mathbf{k} = (k_1, \ldots, k_N)$, $-\infty < k_j < \infty$, and have the form

$$\varphi_\sigma(x) = \beta_\sigma J_{q|n|}(\gamma r) \exp\left(iqn\phi + i\mathbf{k}\mathbf{r}_\parallel - i\omega t\right), \tag{15.2}$$

$$\omega = \sqrt{\gamma^2 + k_m^2}, \quad k_m^2 = k^2 + m^2, \; q = 2\pi/\phi_0, \tag{15.3}$$

where $\mathbf{r}_\parallel = (z_1, \ldots, z_N)$. The eigenvalues for the quantum number γ are quantized by boundary condition (9.11) on the cylindrical surface $r = a$. From this condition it follows that for a given n the possible values of γ are determined by the relation

$$\gamma = \lambda_{n,l}/a, \quad l = 1, 2, \ldots, \tag{15.4}$$

where $\lambda_{n,l}$ are the positive zeros of the function $\bar{J}_{q|n|}(z)$, $\bar{J}_{q|n|}(\lambda_{n,l}) = 0$, arranged in ascending order, $\lambda_{n,l} < \lambda_{n,l+1}$, $n = 0, 1, 2, \ldots$. Here, for a given function $f(z)$, we use the barred notation (3.1) with the coefficients

$$A = \tilde{A}, \; B = -\tilde{B}/a. \tag{15.5}$$

In the following we will assume the values of A/B for which all zeros are real.

The coefficient β_σ in (15.2) is determined from the normalization condition with the integration over the region inside the cylindrical surface and is equal to

$$\beta_\sigma^2 = \frac{\lambda_{n,l} T_{qn}(\lambda_{n,l})}{(2\pi)^N \omega \phi_0 a^2}, \tag{15.6}$$

with the notation $T_\nu(z)$ from (3.8). Substituting the eigenfunctions (15.2) into the mode-sum formula (9.8) with a set of quantum numbers $\sigma = (nl\mathbf{k})$, for the positive frequency Wightman function one finds

$$\begin{aligned} W(x,x') &= 2\int d^N\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{r}_\parallel} \sum_{n=0}^{\infty}{}' \cos(qn\Delta\phi) \\ &\times \sum_{l=1}^{\infty} \beta_\sigma^2 J_{qn}(\gamma r) J_{qn}(\gamma r') e^{-i\omega\Delta t}|_{\gamma=\lambda_{n,l}/a}, \end{aligned} \tag{15.7}$$

where $\Delta\mathbf{r}_\parallel = \mathbf{r}_\parallel - \mathbf{r}'_\parallel$, $\Delta\phi = \phi - \phi'$, $\Delta t = t - t'$, and the prime in the sum means that the summand with $n = 0$ should be taken with the weight 1/2. As we do not know the explicit expressions for the eigenvalues $\lambda_{n,l}$ as functions on n and l, and the summands in the series over l are strongly oscillating functions for large values of l, this formula is not convenient for the further evaluation of the VEVs of the field square and the EMT. In addition, the expression on the right of (15.7) is divergent in the coincidence limit and some renormalization procedure is needed to extract finite result for the VEVs of the field square and the EMT. To obtain an alternative form for the Wightman function we will apply to the sum over l summation formula (3.18). In this formula as a function $f(z)$ we choose

$$f(z) = \frac{zJ_{qn}(zr/a)J_{qn}(zr'/a)}{\sqrt{k_m^2 + z^2/a^2}} \exp(-i\Delta t\sqrt{k_m^2 + z^2/a^2}), \tag{15.8}$$

where $k_m^2 = k^2 + m^2$. The condition to formula (3.18) to be satisfied is $r + r' + |\Delta t| < 2a$. In particular, this is the case in the coincidence limit $t = t'$ for the region under consideration, $r, r' < a$. Formula (3.18) allows to present the Wightman function in the form [43]

$$W(x,x') = W_s(x,x') + \langle\varphi(x)\varphi(x')\rangle_a, \tag{15.9}$$

where

$$\begin{aligned} W_s(x,x') &= \frac{1}{\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} e^{i\mathbf{k}\Delta\mathbf{r}_\parallel} \int_0^\infty dz \frac{ze^{-i\Delta t\sqrt{z^2+k_m^2}}}{\sqrt{z^2+k_m^2}} \\ &\times \sum_{n=0}^{\infty}{}' \cos(qn\Delta\phi) J_{qn}(zr) J_{qn}(zr'), \end{aligned} \tag{15.10}$$

and

$$\begin{aligned} \langle\varphi(x)\varphi(x')\rangle_a &= -\frac{2}{\pi\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} e^{i\mathbf{k}\Delta\mathbf{r}_\parallel} \int_{k_m}^\infty dz \frac{z\cosh(\Delta t\sqrt{z^2-k_m^2})}{\sqrt{z^2-k_m^2}} \\ &\times \sum_{n=0}^{\infty}{}' \cos(qn\Delta\phi) I_{qn}(zr) I_{qn}(zr') \frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)}. \end{aligned} \tag{15.11}$$

In the limit $a \to \infty$ for fixed r, r', the term $\langle\varphi(x)\varphi(x')\rangle_a$ vanishes and, hence, the term $W_s(x,x')$ is the Wightman function for the geometry of a cosmic string without the cylindrical boundary

(the integral representation of the corresponding Green function for a massive scalar field is considered in Refs. [128]). Consequently, the term $\langle\varphi(x)\varphi(x')\rangle_a$ is induced by the presence of the cylindrical boundary. For points away from the cylindrical surface this term is finite in the coincidence limit and the renormalization is needed only for the part coming from (15.10). As it has been shown in Ref. [43], the boundary induced part of the Wightman function in the exterior region, $r > a$, is obtained from the corresponding part in the interior region by the replacements $I \rightleftarrows K$.

15.2 Vacuum expectation values of the field square and the energy-momentum tensor inside a cylindrical shell

Taking the coincidence limit $x' \to x$ in formula (15.9) for the Wightman function and integrating over $\mathbf{k}$, the VEV of the field square is presented as a sum of two terms:

$$\langle 0|\varphi^2|0\rangle = \langle 0_s|\varphi^2|0_s\rangle + \langle\varphi^2\rangle_a, \tag{15.12}$$

where $|0_s\rangle$ is the amplitude for the vacuum state in the geometry when the cylindrical shell is absent. The second term on the right of this formula is induced by the cylindrical boundary and is given by the formula [43]

$$\langle\varphi^2\rangle_a = -\frac{A_D}{\phi_0}\sum_{n=0}^{\infty}{}' \int_m^\infty dz\, z\left(z^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)} I_{qn}^2(zr), \tag{15.13}$$

where we have introduced the notation

$$A_D = \frac{2^{3-D}\pi^{\frac{1-D}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{15.14}$$

The formula for the VEV of the field square in the region outside the cylindrical shell is obtained from (15.13) by the replacements $I \rightleftarrows K$. For points away from the cylindrical surface, $r < a$, the integral in (15.13) is exponentially convergent in the upper limit and the boundary-induced part in the VEV of the field square is finite. In particular, this part is negative for Dirichlet scalar and is positive for Neumann scalar. Near the string, $r \ll a$, the main contribution to $\langle\varphi^2\rangle_a$ comes from the summand with $n = 0$ and one has

$$\langle\varphi^2\rangle_a \approx -\frac{A_D}{2a^{D-1}\phi_0}\int_{ma}^\infty dz\ z\left(z^2 - m^2a^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_0(z)}{\bar{I}_0(z)}. \tag{15.15}$$

As the boundary-free renormalized VEV diverges on the string, we conclude from here that near the string the main contribution to the VEV of the field square comes from this part.

The part $\langle\varphi^2\rangle_a$ diverges on the cylindrical surface $r = a$. Near this surface the main contribution into (15.13) comes from large values of n. Introducing a new integration variable $z \to nqz$, replacing the modified Bessel functions by their uniform asymptotic expansions for large values of the order (see, for instance, [65]), and expanding over $a - r$, to the leading order one finds

$$\langle\varphi^2\rangle_a \approx \frac{(1 - 2\delta_{B0})\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{\frac{D+1}{2}}(a - r)^{D-1}}. \tag{15.16}$$

This leading behavior is the same as that for a cylindrical surface of radius a in the Minkowski spacetime. As the boundary-free part is finite at $r = a$, near the boundary the total renormalized VEV of the field square is dominated by the boundary-induced part and is negative for Dirichlet

scalar. Combining this with the estimation for the region near the string, we come to the conclusion that in this case the VEV of the field square vanishes for some intermediate value of r.

Now we turn to the investigation of the boundary-induced VEV given by (15.13), in the limiting cases of the parameter q. Firstly consider the limit when the parameter q is large which corresponds to small values of ϕ_0 and, hence, to a large planar angle deficit. In this limit the order of the modified Bessel functions for the terms with $n \neq 0$ in (15.13) is large and we can replace these functions by their uniform asymptotic expansions. On the base of these expansions it can be seen that to the leading order the contribution of the terms with $n \neq 0$ is suppressed by the factor $q^{(D-1)/2}(r/a)^{2q}$ and the main contribution to the VEV of the field square comes from the $n = 0$ term:

$$\langle \varphi^2 \rangle_a \approx -\frac{A_D}{2\phi_0} \int_m^\infty dz\, z \left(z^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_0(za)}{\bar{I}_0(za)} I_0^2(zr),\ q \gg 1, \tag{15.17}$$

with the linear dependence on q. In the same limit the boundary-free part in the VEV of the field square behaves as q^{D-1} and, hence, its contribution dominates in comparison with the boundary-induced part. In the opposite limit when $q \to 0$, the series over n in Eq. (15.13) diverges and, hence, for small values of q the main contribution comes from large values n. In this case, to the leading order, we can replace the summation by the integration: $\sum_n f(qn) \to (1/q) \int_0^\infty dx f(x)$. As a consequence, we obtain that in the limit $q \to 0$ the boundary-induced VEV in the field square tends to a finite limiting value:

$$\langle \varphi^2 \rangle_a \approx -\frac{A_D}{2\pi} \int_0^\infty dx \int_m^\infty dz\, z \left(z^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_x(za)}{\bar{I}_x(za)} I_x^2(zr). \tag{15.18}$$

Now we consider the VEV of the EMT for the situation when the cylindrical boundary is present. Similar to the case of the field square, this VEV is written in the form

$$\langle 0|T_{ik}|0\rangle = \langle 0_s|T_{ik}|0_s\rangle + \langle T_{ik}\rangle_a, \tag{15.19}$$

where the part $\langle 0_s|T_{ik}|0_s\rangle$ corresponds to the geometry of a cosmic string without boundaries and $\langle T_{ik}\rangle_a$ is induced by the cylindrical boundary. The first term for a conformally coupled $D = 3$ massless scalar field was evaluated in Ref. [129]. The case of an arbitrary curvature coupling is considered in Refs. [130, 43]. The term induced by the cylindrical shell is obtained from the corresponding part in the Wightman function, acting by the appropriate differential operator and taking the coincidence limit (see formula (9.10)). For points away from the cylindrical surface this limit gives a finite result. For the corresponding components of the EMT one obtains (no summation over i) [43]

$$\langle T_i^i \rangle_a = \frac{A_D}{\phi_0} \sum_{n=0}^{\infty}{}' \int_m^\infty dz z^3 \left(z^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)} F_{qn}^{(i)} \left[I_{qn}(zr)\right], \tag{15.20}$$

with the notations

$$F_{qn}^{(0)}[f(y)] = \left(2\zeta - \frac{1}{2}\right)\left[f'^2(y) + \left(1 + \frac{q^2n^2}{y^2}\right) f^2(y)\right] + \frac{y^2 - m^2r^2}{(D-1)y^2} f^2(y), \tag{15.21}$$

$$F_{qn}^{(1)}[f(y)] = \frac{1}{2} f'^2(y) + \frac{2\zeta}{y} f(y) f'(y) - \frac{1}{2}\left(1 + \frac{q^2n^2}{y^2}\right) f^2(y), \tag{15.22}$$

$$F_{qn}^{(2)}[f(y)] = \left(2\zeta - \frac{1}{2}\right)\left[f'^2(y) + \left(1 + \frac{q^2n^2}{y^2}\right) f^2(y)\right] + \frac{q^2n^2}{y^2} f^2(y) - \frac{2\xi}{y} f(y) f'(y), \tag{15.23}$$

and $F_{qn}^{(i)}[f(y)] = F_{qn}^{(0)}[f(y)]$ for $i = 3, \ldots, D$. The formula for the VEV of the EMT in the region outside the cylindrical shell is obtained from (15.20) by the replacements $I \rightleftarrows K$. It can be checked that the expectation values (15.20) satisfy the continuity equation for the EMT. The boundary-induced part in the VEV of the EMT given by Eq. (15.20) is finite everywhere except at points on the boundary and at points on the string in the case $q < 1$. Unlike to the surface divergences, the divergences on the string are integrable.

In the case $q > 1$, near the string, $r \to 0$, the main contribution to the boundary part (15.20) comes from the summand with $n = 0$ and one has

$$\langle T_i^i \rangle_a \approx \frac{A_D}{2\phi_0 a^{D+1}} \int_{ma}^{\infty} dz z^3 \left(z^2 - m^2 a^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_0(z)}{\bar{I}_0(z)} F^{(i)}(z), \tag{15.24}$$

with the notations

$$F^{(0)}(z) = 2\zeta - \frac{1}{2} + \frac{1 - m^2 a^2/z^2}{D-1}, \; F^{(i)}(z) = \zeta - \frac{1}{2}, \; i = 1, 2. \tag{15.25}$$

For $q < 1$ the main contribution to the boundary-induced part for points near the string comes from $n = 1$ term and in the leading order one has

$$\langle T_i^i \rangle_a \approx \frac{q^2 A_D r^{2q-2} F_1^{(i)}}{2^q \pi \Gamma^2(q+1)} \int_m^{\infty} dz z^{2q+1} \left(z^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{K}_q(za)}{\bar{I}_q(za)}, \tag{15.26}$$

where

$$F_1^{(0)} = q(2\zeta - 1/2), \; F_1^{(1)} = \zeta, \; F_1^{(2)} = (2q-1)\zeta. \tag{15.27}$$

As we see, in this case the VEVs for the EMT diverge on the string. This divergence is integrable. In particular, the corresponding contribution to the energy in the region near the string is finite.

As in the case of the field square, in the limit $q \gg 1$ the contribution of the terms with $n \neq 0$ to the VEV of the EMT is suppressed by the factor $q^{(D-1)/2}(r/a)^{2q}$ and the main contribution comes from the $n = 0$ term with the linear dependence on q. In the same limit, the boundary-free part in the VEV of the EMT behaves as q^{D+1} and, hence, the total energy-momentum tensor is dominated by this part. In the opposite limit, when $q \ll 1$, by the way similar to that used before for the VEV of the field square, it can be seen that the boundary-induced part in the vacuum EMT tends to a finite limiting value which is obtained from (15.20) replacing the summation over n by the integration.

The boundary part $\left\langle T_i^k \right\rangle_a$ diverges on the cylindrical surface $r = a$. Introducing a new integration variable $z \to nqz$ and taking into account that near the surface $r = a$ the main contribution comes from large values of n, we can replace the modified Bessel functions by their uniform asymptotic expansions for large values of the order. To the leading order this gives

$$\langle T_i^i \rangle_a \approx \frac{D(\zeta - \zeta_D)(2\delta_{B0} - 1)}{2^D \pi^{(D+1)/2} (a-r)^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \quad i = 0, 2, \ldots, D. \tag{15.28}$$

This leading divergence does not depend on the parameter q and coincides with the corresponding one for a cylindrical surface of radius a in the Minkowski bulk. For the radial component to the leading order one has $\langle T_1^1 \rangle_a \sim (a-r)^{-D}$. In particular, for a minimally coupled scalar field the corresponding energy density is negative for Dirichlet boundary condition and is positive for non-Dirichlet boundary conditions. For a conformally coupled scalar the leading term vanishes and it is necessary to keep the next term in the corresponding asymptotic expansion. As the boundary-free part in the VEV of the EMT is finite on the cylindrical surface, for points near the boundary the vacuum EMT is dominated by the boundary-induced part. Taking $q = 1$,

from the formulae given in this section we obtain the corresponding results for the geometry of a cylindrical boundary in the Minkowski bulk [41].

The problem considered in this section is closely related to the problem of the investigation of the vacuum densities in the geometry of a wedge with the opening angle ϕ_0 and with the cylindrical boundary of radius a [45, 46]. For a scalar field with Dirichlet boundary condition the corresponding eigenfunctions in the region inside the cylindrical shell are determined by formula (15.2) with the replacement $\exp(iqn\phi) \to \sin(qn\phi)$, $n = 1, 2, \ldots$, and $q = \pi/\phi_0$. The eigenvalues for γ are given by relation (15.4), where now $\lambda_{n,l}$ are the zeros of the function $J_{qn}(x)$. The corresponding Wightman function and the VEVs of the field square and the EMT are evaluated by making use of summation formula (3.18) in the way similar to that used in this section for the geometry of a cosmic string. The corresponding formulae can be found in [46]. Note that in the case of a wedge the VEVs of the field square and the EMT depend on the angle ϕ.

16 Electromagnetic Casimir densities induced by a conducting cylindrical shell in the cosmic string spacetime

In this section we consider the application of the GAPF for the investigation of the polarization of the electromagnetic vacuum by a perfectly conducting cylindrical shell coaxial with the cosmic string [44]. This geometry can be viewed as a simplified model for the superconducting string, in which the string core in what concerns its superconducting effects is taken to be an ideal conductor. The background spacetime is described by the $N = 1$ version of the line element (15.1).

16.1 Vacuum expectation values of the field square inside a cylindrical shell

In the region inside the cylindrical shell we have two different types of the eigenfunctions corresponding to the waves of the electric and magnetic types. In the Coulomb gauge, the vector potentials for these waves are given by the formulae

$$\mathbf{A}_\sigma = \beta_\sigma \left\{ \begin{array}{ll} (1/i\omega)\left(\gamma^2\mathbf{e}_3 + ik\nabla_t\right) J_{q|n|}(\gamma r)\exp\left[i\left(qn\phi + kz - \omega t\right)\right], & \lambda = 0 \\ -\mathbf{e}_3 \times \nabla_t \left\{J_{q|n|}(\gamma r)\exp\left[i\left(qn\phi + kz - \omega t\right)\right]\right\}, & \lambda = 1 \end{array} \right. , \tag{16.1}$$

where $\mathbf{e}_3$ is the unit vector along the cosmic string, ∇_t is the part of the nabla operator transverse to the string, and

$$\omega^2 = \gamma^2 + k^2,\ q = 2\pi/\phi_0,\ n = 0, \pm 1, \pm 2, \ldots. \tag{16.2}$$

Here and in what follows $\lambda = 0$ and $\lambda = 1$ correspond to the cylindrical waves of the electric (transverse magnetic (TM)) and magnetic (transverse electric (TE)) types, respectively. The normalization coefficient in (16.1) is found from the orthonormalization condition (14.3), where the integration goes over the region inside the shell. From this condition, by using the standard integral involving the square of the Bessel function, one finds

$$\beta_\sigma^2 = \frac{qT_{q|n|}(\gamma a)}{\pi\omega a\gamma}, \tag{16.3}$$

with $T_\nu(x)$ defined by (3.8).

The eigenvalues for the quantum number γ are determined by standard boundary conditions (14.5) for the electric and magnetic fields on the cylindrical shell. From boundary conditions we see that these eigenvalues are solutions of the equation

$$J_{q|n|}^{(\lambda)}(\gamma a) = 0, \quad \lambda = 0, 1, \tag{16.4}$$

where a is the radius of the cylindrical shell, $J_\nu^{(0)}(x) = J_\nu(x)$ and $J_\nu^{(1)}(x) = J_\nu'(x)$. We will denote the corresponding eigenmodes by $\gamma a = j_{n,l}^{(\lambda)}$, $l = 1, 2, \ldots$. As a result the eigenfunctions are specified by the set of quantum numbers $\sigma = (k, n, \lambda, l)$.

Substituting the eigenfunctions into the corresponding mode-sum formula, for the VEVs of the squares of the electric and magnetic fields inside the shell we find

$$\begin{aligned} \langle 0|F^2|0\rangle &= \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{F}_\sigma^* = \frac{2q}{\pi a^3} \sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \sum_{\lambda=0,1} \sum_{l=1}^{\infty} j_{n,l}^{(\lambda)3} \\ &\times \frac{T_{qn}(j_{n,l}^{(\lambda)})}{\sqrt{j_{n,l}^{(\lambda)2} + k^2 a^2}} g_{qn}^{(\eta_{F\lambda})}[k, J_{qn}(j_{n,l}^{(\lambda)} r/a)], \end{aligned} \tag{16.5}$$

where $F = E, B$ with $\eta_{E\lambda} = \lambda$, $\eta_{B\lambda} = 1 - \lambda$, and the prime in the summation means that the term $n = 0$ should be halved. In (16.5), for a given function $f(x)$, we have used the notations

$$g_\nu^{(j)}[k, f(x)] = \begin{cases} (k^2 r^2/x^2)\left[f'^2(x) + \nu^2 f^2(x)/x^2\right] + f^2(x), & j = 0 \\ (1 + k^2 r^2/x^2)\left[f'^2(x) + \nu^2 f^2(x)/x^2\right], & j = 1 \end{cases}. \tag{16.6}$$

The expressions (16.5) corresponding to the electric and magnetic fields are divergent. They may be regularized introducing a cutoff function $\psi_\mu(\omega)$ with the cutting parameter μ which makes the divergent expressions finite and satisfies the condition $\psi_\mu(\omega) \to 1$ for $\mu \to 0$. After the renormalization the cutoff function is removed by taking the limit $\mu \to 0$. An alternative way is to consider the product of the fields at different spacetime points and to take the coincidence limit after the subtraction of the corresponding Minkowskian part. Here we will follow the first approach.

In order to evaluate the mode-sum in (16.5), we apply to the series over j summation formula (3.18). As it can be seen, for points away from the shell the contribution to the VEVs coming from the second integral term on the right-hand side of (3.18) is finite in the limit $\mu \to 0$ and, hence, the cutoff function in this term can be safely removed. As a result the VEVs can be written as

$$\langle 0|F^2|0\rangle = \langle 0_s|F^2|0_s\rangle + \langle F^2\rangle_a, \tag{16.7}$$

where

$$\begin{aligned} \langle 0_s|F^2|0_s\rangle &= \frac{q}{\pi} \sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \int_0^\infty d\gamma\, \frac{\gamma^3 \psi_\mu(\omega)}{\sqrt{\gamma^2 + k^2}} \\ &\times \left[\left(1 + 2\frac{k^2}{\gamma^2}\right)\left(J_{qn}'^2(\gamma r) + \frac{q^2 n^2}{\gamma^2 r^2} J_{qn}^2(\gamma r)\right) + J_{qn}^2(\gamma r)\right], \end{aligned} \tag{16.8}$$

and

$$\langle F^2\rangle_a = \frac{4q}{\pi^2} \sum_{n=0}^{\infty}{}' \int_0^\infty dk \sum_{\lambda=0,1} \int_k^\infty dx\, x^3\, \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} \frac{G_{qn}^{(\eta_{F\lambda})}[k, I_{qn}(xr)]}{\sqrt{x^2 - k^2}}. \tag{16.9}$$

Note that in Eq. (16.9) we used the notations

$$G_\nu^{(j)}[k, f(x)] = \begin{cases} (k^2 r^2/x^2)\left[f'^2(x) + \nu^2 f^2(x)/x^2\right] + f^2(x), & j = 0 \\ (k^2 r^2/x^2 - 1)\left[f'^2(x) + \nu^2 f^2(x)/x^2\right], & j = 1 \end{cases}. \tag{16.10}$$

The second term on the right-hand side of Eq. (16.7) vanishes in the limit $a \to \infty$. Thus, we can conclude that the term $\langle 0_s|F^2|0_s\rangle$ corresponds to the part in VEVs when the cylindrical

shell is absent with the corresponding vacuum state $|0_s\rangle$. Note that for the geometry without boundaries one has $\langle 0_s|E^2|0_s\rangle = \langle 0_s|B^2|0_s\rangle$. Hence, the application of the GAPF enables us to extract from the VEVs the boundary-free parts and to write the boundary-induced parts in terms of the exponentially convergent integrals. The boundary-free parts can be further simplified with the final result

$$\langle F^2\rangle_{s,\mathrm{ren}} = -\frac{(q^2-1)(q^2+11)}{180\pi r^4}. \tag{16.11}$$

Changing the integration variable to $y=\sqrt{x^2-k^2}$ and introducing polar coordinates in the (k,y) plane, after the explicit integration over the angular part, the part in the VEV induced by the cylindrical shell can be written in the form [44]

$$\langle F^2\rangle_a = \frac{q}{\pi}{\sum_{n=0}^{\infty}}' \sum_{\lambda=0,1}\int_0^\infty dx\, x^3 \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} G_{qn}^{(\eta_{F\lambda})}\left[I_{qn}(xr)\right], \tag{16.12}$$

where we have used the notation

$$G_\nu^{(j)}\left[f(x)\right] = \left\{ \begin{array}{cc} f'^2(x)+\nu^2 f^2(x)/x^2+2f^2(x), & j=0 \\ -f'^2(x)-\nu^2 f^2(x)/x^2, & j=1 \end{array} \right. . \tag{16.13}$$

The boundary-induced parts for the electric and magnetic fields are different and, hence, the presence of the shell breaks the electric-magnetic symmetry in the VEVs. Of course, this is a consequence of different boundary conditions for the electric and magnetic fields. The formulae for the VEVs of the field squares in the region outside the cylindrical shell are obtained from (16.12) by the replacements $I \rightleftarrows K$.

The expression in the right-hand side of (16.12) is finite for $0<r<a$ and diverges on the shell with the leading term $\langle E^2\rangle_a \approx -\langle B^2\rangle_a \approx (3/4\pi)(a-r)^{-4}$. Near the string, $r/a \ll 1$, the asymptotic behavior of the boundary induced part in the VEVs of the field squares depends on the parameter q. For $q \geqslant 1$, the dominant contribution comes from the lowest mode $n=0$ and to the leading order one has $\langle E^2\rangle_a \approx 0.32q/a^4$, $\langle B^2\rangle_a \approx -0.742q/a^4$. For $q<1$ the main contribution comes form the mode with $n=1$ and the boundary-induced parts diverge on the string. The leading terms are given by

$$\langle E^2\rangle_a \approx -\langle B^2\rangle_a \approx \frac{(r/a)^{2(q-1)}}{2^{2(q-1)}\pi\Gamma^2(q)a^4}\int_0^\infty dx\, x^{2q+1}\left[\frac{K_q(x)}{I_q(x)} - \frac{K_q'(x)}{I_q'(x)}\right]. \tag{16.14}$$

As for points near the shell, here the leading divergence is cancelled in the evaluation of the vacuum energy density. In accordance with (16.11), near the string the total VEV is dominated by the boundary-free part. Here we have considered the VEV for the field square. The VEVs for the bilinear products of the fields at different spacetime points may be evaluated in a similar way.

Now, we turn to the investigation of the behavior of the boundary-induced VEVs in the asymptotic regions of the parameter q. For small values of this parameter, $q \ll 1$, the main contribution into (16.12) comes from large values of n. In this case, we can replace the summation over n by an integration in accordance with the correspondence

$${\sum_{n=0}^{\infty}}' h(qn) \to \frac{1}{q}\int_0^\infty dx\, h(x). \tag{16.15}$$

By making this replacement, we can see from (16.12) that, in this situation, the boundary induced VEVs tend to a finite value. Note that the same is the case for the boundary-free part

(16.11). In the limit $q \gg 1$, the order of the modified Bessel functions is large for $n \neq 0$. By using the corresponding asymptotic formulae it can be seen that the contribution of these term is suppressed by the factor $\exp[-2qn\ln(a/r)]$. As a result, the main contribution comes from the lowest mode $n = 0$ and the boundary induced VEVs behave like q.

16.2 Vacuum expectation value for the energy-momentum tensor

In this subsection we consider the vacuum EMT in the region inside the cylindrical shell. Substituting the eigenfunctions (16.1) into the corresponding mode-sum formula, we obtain (no summation over i)

$$\langle 0|T_i^k|0\rangle = \frac{q\delta_i^k}{4\pi^2 a^3} \sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \sum_{\lambda=0,1} \sum_{l=1}^{\infty} \frac{j_{n,l}^{(\lambda)3} T_{qn}(j_{n,l}^{(\lambda)})}{\sqrt{j_{n,l}^{(\lambda)2} + k^2 a^2}} f_{qn}^{(i)}[k, J_{qn}(j_{n,l}^{(\lambda)} r/a)], \tag{16.16}$$

where we have introduced the notations

$$\begin{aligned} f_\nu^{(i)}[k, f(x)] &= (-1)^i \left(2k^2/\gamma^2 + 1\right) \left[f'^2(x) + \nu^2 f^2(x)/x^2\right] + f^2(x), && (16.17)\\ f_\nu^{(j)}[k, f(x)] &= (-1)^i f'^2(x) - \left[1 + (-1)^i \nu^2/x^2\right] f^2(x), && (16.18) \end{aligned}$$

with $i = 0, 3$ and $j = 1, 2$. As in the case of the field square, we apply summation formula (3.18) to rewrite the sum over l. This enables us to present the VEV as the sum of boundary-free and boundary-induced parts as follows

$$\langle 0|T_i^k|0\rangle = \langle 0_s|T_i^k|0_s\rangle + \langle T_i^k\rangle_a. \tag{16.19}$$

The part induced by the cylindrical shell may be written in the form (no summation over i) [44]

$$\langle T_i^k\rangle_a = \frac{q\delta_i^k}{4\pi^2} \sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1} \int_0^{\infty} dx x^3 \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} F_{qn}^{(i)}[I_{qn}(xr)], \tag{16.20}$$

with the notations

$$\begin{aligned} F_\nu^{(0)}[f(y)] &= F_\nu^{(3)}[f(y)] = f^2(y), && (16.21)\\ F_\nu^{(i)}[f(y)] &= -(-1)^i f'^2(y) - \left[1 - (-1)^i \nu^2/y^2\right] f^2(y),\ i = 1, 2. && (16.22) \end{aligned}$$

As it can be easily checked, this tensor is traceless and satisfies the covariant continuity equation $\nabla_k \langle T_i^k\rangle_a = 0$. By using the inequalities $[I_\nu(y)K_\nu(y)]' < 0$ and $I'_\nu(y) < \sqrt{1 + \nu^2/y^2} I_\nu(y)$, and the recurrence relations for the modified Bessel functions, it can be seen that the boundary-induced parts in the vacuum energy density and axial stress are negative, whereas the corresponding radial and azimuthal stresses are positive. The VEV of the EMT in the region outside the cylindrical shell is obtained from (16.20) by the replacements $I \rightleftarrows K$.

The renormalized VEV of the EMT for the geometry without the cylindrical shell is obtained by using the corresponding formulae for the field square (16.11). For the corresponding energy density one finds [130]

$$\langle T_0^0\rangle_{s,\mathrm{ren}} = -\frac{(q^2 - 1)(q^2 + 11)}{720\pi^2 r^4}. \tag{16.23}$$

Other components are found from the tracelessness condition and the continuity equation.

Now, let us discuss the behavior of the boundary-induced part in the VEV of the EMT in the asymptotic region of the parameters. Near the cylindrical shell the main contribution comes

from large values of n. Thus, using the uniform asymptotic expansions for the modified Bessel functions for large values of the order, up to the leading order, we find

$$\langle T_0^0 \rangle_{\rm b} \approx -\frac{1}{2}\langle T_2^2 \rangle_{\rm b} \approx -\frac{(a-r)^{-3}}{60\pi^2 a},\ \langle T_1^1 \rangle_{\rm b} \approx \frac{(a-r)^{-2}}{60\pi^2 a^2}. \tag{16.24}$$

These leading terms do not depend on the planar angle deficit in the cosmic string geometry. Near the cosmic string the main contribution comes from the mode $n = 0$ and we have

$$\langle T_0^0 \rangle_{\rm b} \approx -\langle T_1^1 \rangle_{\rm b} \approx -\langle T_2^2 \rangle_{\rm b} \approx \frac{q}{8\pi^2 a^4} \int_0^\infty dx\, x^3 \left[\frac{K_0(x)}{I_0(x)} - \frac{K_1(x)}{I_1(x)} \right] = -0.0168 \frac{q}{a^4}. \tag{16.25}$$

Therefore, differently from the VEV for the field square, the boundary-induced part in the vacuum EMT is finite on the string for all values of q.

The behavior of the boundary-induced part in the VEV of the EMT in the asymptotic regions of the parameter q is investigated in a way analogous to that for the field square. For $q \ll 1$, we replace the summation over n by the integration in accordance with (16.15) and the VEV tends to a finite limiting value which does not depend on q. Note that as the spatial volume element is proportional to $1/q$, in this limit the global quantities such as the integrated vacuum energy behave as $1/q$. In the limit $q \gg 1$, the contribution of the modes with $n \geqslant 1$ is suppressed by the factor $\exp[-2qn\ln(a/r)]$ and the main contribution comes from the $n = 0$ mode with the behavior $\propto q$. Though in this limit the vacuum densities are large, due to the factor $1/q$ in the spatial volume the corresponding global quantities tend to finite value. The VEVs for the EMT of the electromagnetic field induced by perfectly conducting cylindrical shell in the Minkowski spacetime considered in [39], are obtained from the formulae given in this section taking $q = 1$.

The VEVs of the field square and the EMT for electromagnetic field inside a perfectly conducting wedge with the opening angle ϕ_0 and with a conducting cylindrical boundary of radius a can be evaluated by the way similar to that used in this section for the geometry of a cosmic string. The corresponding results are given in [47]. In the geometry of a wedge the eigenfunctions for the vector potential in the region inside the cylindrical shell are given by formula (16.1) with the replacements $\exp(iqn\phi) \to \sin(qn\phi)$, $n = 1, 2, \ldots$, for TM modes ($\lambda = 0$) and $\exp(iqn\phi) \to \cos(qn\phi)$, $n = 1, 2, \ldots$, for TE modes ($\lambda = 1$), where now $q = \pi/\phi_0$. In the case of a wedge the VEVs of the field square and the EMT depend on the angle ϕ. The vacuum energy density induced by the cylindrical shell is negative for the interior region and the corresponding vacuum forces acting on the wedge sides are always attractive.

17 Vacuum densities in the region between two coaxial cylindrical surfaces

17.1 Scalar field

In this subsection, we consider the positive frequency Wightman function, the VEVs of the field square and the EMT for a massive scalar field with general curvature coupling parameter in the region between two coaxial cylindrical surfaces with radii a and b, $a < b$, on background of the $(D+1)$-dimensional Minkowski spacetime [42]. In an appropriately chosen cylindrical system of coordinates the corresponding line element has the form (15.1) with $0 \leqslant \phi \leqslant 2\pi$. We will assume that on the bounding surfaces the field obeys the boundary conditions

$$\left(\tilde{A}_j + \tilde{B}_j n^i_{(j)} \nabla_i \right) \varphi(x) \big|_{r=j} = 0,\ j = a, b, \tag{17.1}$$

with $\tilde{A}_j$ and $\tilde{B}_j$ being constants, $n^i_{(j)}$ is the inward-pointing normal to the bounding surface $r = j$. For the region between the surfaces, $a \leqslant r \leqslant b$, one has $n^i_{(j)} = n_j \delta^i_1$ with the notations

$n_a = 1$ and $n_b = -1$. The eigenfunctions are specified by the set of quantum numbers $\sigma = (\gamma, n, \mathbf{k})$, $n = 0, \pm 1, \pm 2, \ldots$, and have the form

$$\varphi_\sigma(x) = \beta_\sigma g_{|n|}(\gamma a, \gamma r) \exp(in\phi + i\mathbf{k}\mathbf{r}_\parallel - i\omega t), \tag{17.2}$$

with

$$g_n(\gamma a, \gamma r) = \bar{Y}_n^{(a)}(\gamma a) J_n(\gamma r) - \bar{J}_n^{(a)}(\gamma a) Y_n(\gamma r), \tag{17.3}$$

and the other notations are the same as in Section 15. The barred notation is defined by formula (4.2) with the coefficients

$$A_j = \tilde{A}_j,\ B_j = n_j \tilde{B}_j / j,\ \ j = a, b. \tag{17.4}$$

From the boundary condition on the surface $r = b$ it follows that the possible values of γ are solutions to the equation ($\eta = b/a$)

$$C_n^{ab}(\eta, \gamma a) \equiv \bar{J}_n^{(a)}(\gamma a) \bar{Y}_n^{(b)}(\gamma b) - \bar{Y}_n^{(a)}(\gamma a) \bar{J}_n^{(b)}(\gamma b) = 0. \tag{17.5}$$

The corresponding positive roots we will denote by $\gamma a = \gamma_{n,l}$, $l = 1, 2, \ldots$, assuming that they are arranged in the ascending order, $\gamma_{n,l} < \gamma_{n,l+1}$.

From the orthonormality condition for the eigenfunctions, for the coefficient β_σ one finds

$$\beta_\sigma^2 = \frac{\pi^2 \gamma T_n^{ab}(\gamma a)}{4\omega a (2\pi)^{D-1}}, \tag{17.6}$$

with the notation from (4.7). Substituting eigenfunctions into the mode-sum formula (9.8), for the positive frequency Wightman function one finds

$$\begin{aligned} W(x, x') &= \frac{\pi^2}{2a} \int d^N\mathbf{k} \sideset{}{'}\sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{z g_n(z, zr/a) g_n(z, zr'/a)}{(2\pi)^{D-1}\sqrt{z + k_m^2 a^2}} \\ &\quad \times T_n^{ab}(z) \cos(n\Delta\phi) \exp(i\mathbf{k}\Delta\mathbf{r}_\parallel - i\omega\Delta t)\big|_{z=\gamma_{n,l}}, \end{aligned} \tag{17.7}$$

where, as before, $k_m^2 = k^2 + m^2$ and the prime on the summation sign means that the summand with $n = 0$ should be halved. For the further evaluation of this VEV we apply to the sum over l summation formula (4.14) with

$$h(x) = \frac{x g_n(x, xr/a) g_n(x, xr'/a)}{\sqrt{x^2 + k_m^2 a^2}} \exp(-i\Delta t \sqrt{x^2/a^2 + k_m^2}). \tag{17.8}$$

The corresponding conditions are satisfied if $r + r' + |\Delta t| < 2b$. In particular, this is the case in the coincidence limit $t = t'$ for the region under consideration. Now we can see that the application of formula (4.14) allows to present the Wightman function in the form

$$\begin{aligned} W(x, x') &= \frac{1}{(2\pi)^{D-1}} \sideset{}{'}\sum_{n=0}^{\infty} \cos(n\Delta\phi) \int d^N\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{r}_\parallel} \\ &\quad \times \Bigg[\frac{1}{a} \int_0^\infty dz \frac{h(z)}{\bar{J}_n^{(a)2}(z) + \bar{Y}_n^{(a)2}(z)} - \frac{2}{\pi} \int_{k_m}^\infty dz \frac{x\Omega_{a\nu}(az, bz)}{\sqrt{z^2 - k_m^2}} \\ &\quad \times G_n^{(a)}(az, zr) G_n^{(a)}(az, zr') \cosh(\Delta t \sqrt{z^2 - k_m^2}) \Bigg], \end{aligned} \tag{17.9}$$

with the notations from (12.30).

In the limit $b\to\infty$ the second term in figure braces on the right of (17.9) vanishes, whereas the first term does not depend on b. It follows from here that the part with the first term presents the Wightman function in the region outside of a single cylindrical shell with radius a. As a result the Wightman function is presented in the form [42]

$$\begin{aligned}
W(x,x') &= W_{\mathrm{M}}(x,x') + \langle\varphi(x)\varphi(x')\rangle_a \\
&\quad -\frac{2^{2-D}}{\pi^D}\sum_{n=0}^{\infty}{}'\cos(n\Delta\phi)\int d^N\mathbf{k}\,e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_{k_m}^{\infty}dz\,z\frac{\Omega_{an}(az,bz)}{\sqrt{z^2-k_m^2}} \\
&\quad \times G_n^{(a)}(az,zr)G_n^{(a)}(az,zr')\cosh(\Delta t\sqrt{z^2-k_m^2}),
\end{aligned} \tag{17.10}$$

where $W_{\mathrm{M}}(x,x')$ is the Wightman function for a scalar field in the unbounded Minkowskian spacetime, and

$$\begin{aligned}
\langle\varphi(x)\varphi(x')\rangle_a &= -\frac{2^{2-D}}{\pi^D}\sum_{n=0}^{\infty}{}'\cos(n\Delta\phi)\int d^N\mathbf{k}\,e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_{k_m}^{\infty}dz\,z \\
&\quad \times\frac{\bar{I}_n^{(a)}(az)}{\bar{K}_n^{(a)}(az)}\frac{K_n(zr)K_n(zr')}{\sqrt{z^2-k_m^2}}\cosh(\Delta t\sqrt{z^2-k_m^2}),
\end{aligned} \tag{17.11}$$

is the part of the Wightman function induced by a single cylindrical shell with radius a in the region $r>a$. Hence, the last term on the right of (17.10) is induced by the presence of the second shell with radius b. It can be seen that the Wightman function can also be presented in an equivalent form which is obtained from (17.10) by the replacement $a\to b$ except in the argument of the function $\Omega_{an}(az,bz)$. In this representation

$$\begin{aligned}
\langle\varphi(x)\varphi(x')\rangle_b &= -\frac{2^{2-D}}{\pi^D}\sum_{n=0}^{\infty}{}'\cos(n\Delta\phi)\int d^N\mathbf{k}\,e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_{k_m}^{\infty}dz\,z \\
&\quad \times\frac{\bar{K}_n^{(b)}(bz)}{\bar{I}_n^{(b)}(bz)}\frac{I_n(zr)I_n(zr')}{\sqrt{z^2-k_m^2}}\cosh(\Delta t\sqrt{z^2-k_m^2}),
\end{aligned} \tag{17.12}$$

is the part induced by a single cylindrical shell with radius b in the region $r<b$. This formula is also directly obtained from (15.11) taking $\phi_0=2\pi$.

By making use of the formulae for the Wightman function and taking the coincidence limit of the arguments, for the VEV of the field square one finds

$$\begin{aligned}
\langle 0|\varphi^2|0\rangle &= \langle\varphi^2\rangle_{\mathrm{M}} + \langle\varphi^2\rangle_j - B_D\sum_{n=0}^{\infty}{}'\int_m^{\infty}du\,u \\
&\quad \times\left(u^2-m^2\right)^{\frac{D-3}{2}}\Omega_{jn}(au,bu)G_n^{(j)2}(ju,ru),
\end{aligned} \tag{17.13}$$

where $j=a$ and $j=b$ provide two equivalent representations and

$$B_D = \frac{2^{2-D}}{\pi^{\frac{D+1}{2}}\Gamma\left(\frac{D-1}{2}\right)}. \tag{17.14}$$

For points away from the boundaries the last two terms on the right of formula (17.13) are finite and, hence, the subtraction of the Minkowskian part without boundaries is sufficient to obtain the renormalized value for the VEV: $\langle\varphi^2\rangle_{\mathrm{ren}}=\langle 0|\varphi^2|0\rangle-\langle\varphi^2\rangle_{\mathrm{M}}$. In formula (17.13) the part $\langle\varphi^2\rangle_j$ is induced by a single cylindrical surface with radius j when the second surface is absent.

The formulae for these terms are obtained from the Wightman function in the coincidence limit. For $j = a$ one has

$$\langle\varphi^2\rangle_a = -B_D \sum_{n=0}^{\infty}{}' \int_m^\infty du\, u \left(u^2 - m^2\right)^{\frac{D-3}{2}} \frac{\bar{I}_n^{(a)}(au)}{\bar{K}_n^{(a)}(au)} K_n^2(ru), \tag{17.15}$$

and the formula for $\langle\varphi^2\rangle_b$ is obtained from here by the replacements $a \to b$, $I \rightleftarrows K$. The last term on the right of formula (17.13) is induced by the presence of the second cylindrical surface.

The VEV for the EMT is obtained by using the formulae for the Wightman function and the VEV of the field square:

$$\begin{aligned}\langle 0|T_i^k|0\rangle &= \langle T_i^k\rangle^{(0)} + \langle T_i^k\rangle_j + B_D\delta_i^k \sum_{n=0}^{\infty}{}' \int_m^\infty du\, u^3 \\ &\quad \times \left(u^2 - m^2\right)^{\frac{D-3}{2}} \Omega_{jn}(au, bu) F_n^{(i)}[G_n^{(j)}(ju, ru)],\end{aligned} \tag{17.16}$$

where the notations $F_\nu^{(i)}[f(z)]$ are defined by formulae (15.21)-(15.23) with $f(z) = G_n^{(j)}(ju, z)$ and $F_n^{(i)}[f(z)] = F_n^{(0)}[f(z)]$ for $i = 3, \ldots, D-1$. In formula (17.16), the term $\langle T_i^k\rangle^{(j)}$ is induced by a single cylindrical surface with radius j. These parts for both interior and exterior regions are investigated in [41] and are obtained from formulae of subsection 15.2 taking $\phi_0 = 2\pi$. The formula for the case $j = a$ is obtained from (17.15) by the replacement $K_n^2(ru) \to u^2 F_n^{(i)}[K_n(ru)]$.

The vacuum force per unit surface of the cylinder at $r = j$ is determined by the $\frac{1}{1}-$component of the vacuum EMT at this point. Similar to the case of spherical geometry, for the region between two surfaces the corresponding effective pressures can be presented as the sum of self-action and interaction parts, formula (12.38). The interaction forces are determined from the last term on the right of formula (17.16) with $i = k = 1$ taking $r = j$:

$$\begin{aligned}p_{(\mathrm{int})}^{(j)} &= \frac{A_D}{2j^2} \sum_{n=0}^{\infty}{}' \int_m^\infty du\, u \left(u^2 - m^2\right)^{\frac{D-3}{2}} \Omega_{jn}(au, bu) \\ &\quad \times \left[\left(n^2/j^2 + u^2\right) B_j^2 + 4\xi n_j A_j B_j/j - A_j^2\right].\end{aligned} \tag{17.17}$$

The expression on the right of this formula is finite for all non-zero distances between the shells. It can be seen that the vacuum effective pressures are negative for both Dirichlet and Neumann scalars and, hence, the corresponding interaction forces are attractive. For the general Robin case the interaction force can be either attractive or repulsive in dependence on the coefficients in the boundary conditions. By using the properties of the modified Bessel functions, the interaction forces per unit surface can also be presented in another equivalent form [42]:

$$\begin{aligned}p_{(\mathrm{int})}^{(j)} &= \frac{A_D n_j}{2j} \sum_{n=0}^{\infty}{}' \int_m^\infty du\, u \left(u^2 - m^2\right)^{\frac{D-3}{2}} \\ &\quad \times \left[1 + (4\xi - 1)\frac{n_j A_j B_j}{jB_{jn}(u)}\right] \frac{\partial}{\partial j} \ln\left|1 - \frac{\bar{I}_n^{(a)}(au)\bar{K}_n^{(b)}(bu)}{\bar{I}_n^{(b)}(bu)\bar{K}_n^{(a)}(au)}\right|.\end{aligned} \tag{17.18}$$

The relation between the interaction forces and the corresponding bulk and surface Casimir energies in the geometry under consideration is discussed in [42].

17.2 Electromagnetic field

In this subsection we consider the VEV for the EMT of the electromagnetic field in the region between two coaxial cylindrical surfaces with radii a and b, $a < b$ [40]. The corresponding

eigenfunctions satisfying the boundary condition on the surface $r = a$ have the form

$$\mathbf{A}_\sigma = \beta_\sigma \left\{ \begin{array}{ll} (1/i\omega)\left(\gamma^2\mathbf{e}_3 + ik\nabla_t\right) P_{\lambda n}(\gamma a, \gamma r) \exp\left[i\left(n\phi + kz - \omega t\right)\right], & \lambda = 0 \\ -\mathbf{e}_3 \times \nabla_t \left\{P_{\lambda n}(\gamma a, \gamma r) \exp\left[i\left(n\phi + kz - \omega t\right)\right]\right\}, & \lambda = 1 \end{array} \right., \tag{17.19}$$

where

$$P_{\lambda n}(x, y) = J_n(y)Y_n^{(\lambda)}(x) - Y_n(y)J_n^{(\lambda)}(x), \tag{17.20}$$

and the other notations are the same as in (16.1). From the boundary conditions on $r = b$ one obtains that the eigenvalues for γ have to be solutions to the following equations

$$\partial_r^\lambda P_{\lambda n}(\gamma a, \gamma r)|_{r=b} = 0. \tag{17.21}$$

These equations have an infinite number of simple real solutions. The eigenvalue equation (17.21) can be written in terms of function (4.1) as

$$C_n^{ab}(\eta, \gamma b) = 0, \quad A_a = A_b = 1 - \lambda,\ B_a = B_b = \lambda, \quad \lambda = 0, 1 \tag{17.22}$$

(see notation (3.1)). Now the normalization coefficient β_σ is given by the formulae

$$\beta_\sigma^2 = \frac{\pi}{4\omega} \left\{ \begin{array}{ll} \left[J_n^2(z)/J_n^2(z\eta) - 1\right]^{-1}, & \lambda = 0 \\ \left[\left(1 - n^2/z^2\eta^2\right) J_n'^2(z)/J_n'^2(z\eta) - 1 + n^2/z^2\right]^{-1}, & \lambda = 1 \end{array} \right., \tag{17.23}$$

where $z = \gamma a$, $\eta = b/a$. Note that this coefficients can be expressed in terms of function (4.7) as

$$\beta_\alpha^2 = \frac{\pi z^{2\lambda - 1}}{4\omega} T_n^{ab}(\eta, z). \tag{17.24}$$

Using these relations and introducing the cutoff function ψ_μ, the vacuum EMT can be written in the form of the following finite integrosums (no summation over i)

$$\begin{aligned} \langle 0|T_i^k|0\rangle &= \frac{\delta_i^k}{16a^3} \sideset{}{'}\sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dk \sum_{\lambda=0,1} \sum_{l=1}^{\infty} \frac{\psi_\mu(\gamma_{n,l}^{(\lambda)}/a)}{\sqrt{\gamma_{n,l}^{(\lambda)2} + k^2a^2}} \\ & \times T_m^{ab}(\eta, \gamma_{n,l}^{(\lambda)}) \gamma_{n,l}^{(\lambda)3+2\lambda} f_n^{(i)}[k, P_{\lambda n}(\gamma_{n,l}^{(\lambda)}, \gamma_{n,l}^{(\lambda)} r/a)], \end{aligned} \tag{17.25}$$

where $\gamma a = \gamma_{n,l}^{(\lambda)}$, $l = 1, 2, \ldots$, are the solutions to eigenvalue equations (17.21) and the expressions for the functions $f_n^{(i)}[k, P_{\lambda n}(\gamma a, x)]$ are obtained from (16.17) and (16.18) taking $f(x) = P_{\lambda n}(\gamma a, x)$. By choosing in formula (4.14)

$$h(z) = \frac{z^{3+2\lambda}\psi_\mu(z/a)}{\sqrt{z^2 + k^2a^2}} f_n^{(i)}[k, P_{\lambda n}(z, zx)], \tag{17.26}$$

one obtains that the VEV of the electromagnetic EMT in the region between two coaxial conducting cylindrical surfaces is presented in the form [40]

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle^{(a)} + \langle T_i^k\rangle_{ab}, \quad a < r < b. \tag{17.27}$$

In this formula the first term on the right is given by (no summation over i)

$$\langle 0|T_i^k|0\rangle^{(a)} = \frac{\delta_i^k}{8\pi^2} \sideset{}{'}\sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dk \int_0^\infty dz \sum_{\lambda=0,1} \frac{z^3\psi_\mu(z)}{\sqrt{k^2 + z^2}} \frac{f_n^{(i)}[k, P_{\lambda n}(az, rz)]}{J_\nu^{(\lambda)2}(az) + Y_\nu^{(\lambda)2}(az)}, \tag{17.28}$$

and for the second term we have

$$\langle T_i^k \rangle_{ab} = \frac{\delta_i^k}{4\pi^2} \sum_{n=0}^{\infty}{}' \int_0^\infty dz \sum_{\lambda=0,1} z^3 \frac{F_n^{(i)}[Q_{\lambda n}(az, rz)] K_n^{(\lambda)}(bz)/K_n^{(\lambda)}(az)}{K_n^{(\lambda)}(az) I_n^{(\lambda)}(bz) - K_n^{(\lambda)}(bz) I_n^{(\lambda)}(az)}, \tag{17.29}$$

with the notation $F_\nu^{(i)}[f(y)]$ from (16.21), (16.22) and

$$Q_{\lambda n}(z, y) = K_n^{(\lambda)}(z) I_n(y) - I_n^{(\lambda)}(z) K_n(y). \tag{17.30}$$

Note that the part (17.29) is finite for $a \leqslant r < b$ and in this part we have removed the cutoff function.

In the limit $b \to \infty$ the second term on the right of (17.27) tends to zero, whereas the first one does not depend on b. From here we conclude that the term $\langle 0|T_i^k|0\rangle^{(a)}$ is the VEV in the region outside a single cylindrical shell with radius a. For the renormalization of this term we subtract the contribution of unbounded Minkowski spacetime which can be presented in the form:

$$\langle 0|T_i^k|0\rangle_{\mathrm{M}} = \frac{\delta_i^k}{4\pi^2} \sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \int_0^\infty dz \frac{z^3 \psi_\mu(z)}{\sqrt{k^2 + z^2}} f_n^{(i)}[k, J_n(rz)]. \tag{17.31}$$

By using the identity

$$\frac{f_n^{(i)}[k, P_{\lambda n}(az, rz)]}{J_\nu^{(\lambda)2}(az) + Y_\nu^{(\lambda)2}(az)} - f_n^{(i)}[k, J_n(rz)] = -\frac{1}{2} \sum_{s=1,2} \frac{J_n^{(\lambda)}(az)}{H_n^{(s)(\lambda)}(az)} f_n^{(i)}[k, H_n^{(s)}(rz)], \tag{17.32}$$

and rotating the integration contour for z by angle $\pi/2$ for $s = 1$ and by angle $-\pi/2$ for $s = 2$, for the renomalized components we obtain

$$\langle T_i^k \rangle_a = \frac{\delta_i^k}{4\pi^2} \sum_{n=0}^{\infty}{}' \int_0^\infty dz\, z^3 \left[\frac{I_n(az)}{K_n(az)} + \frac{I_n'(az)}{K_n'(az)} \right] F_n^{(i)}[K_n(rz)]. \tag{17.33}$$

Note that the corresponding expressions in the region inside a single cylindrical shell are obtained from (16.20) taking $q = 1$ and differ from (17.33) by the replacements $I \rightleftarrows K$. As in the case of the interior components, the renormalized VEV (17.33) is divergent when $r \to a$ with the leading terms given by formulae (16.24). The corresponding asymptotic behavior at large distances from the cylinder, $r \gg a$, can be found from (17.33) introducing a new integration variable $y = rz$ and expanding the integrands over a/r. In this limit the main contribution comes from the lowest order mode with $n = 0$ and one obtains (no summation over i)

$$\langle T_i^k \rangle_a \approx \frac{\delta_i^k c^{(i)}}{8\pi^2 r^4 \ln(r/a)}, \quad c^{(0)} = c^{(1)} = \frac{1}{3},\ c^{(2)} = -1, \quad r \gg a. \tag{17.34}$$

Here, compared to the spherical case, the corresponding quantities tend to zero more slowly.

From the continuity equation for the vacuum EMT one has the following integral relation

$$\langle T_1^1 \rangle_a = \frac{2}{r^2} \int_r^\infty dr\, r \langle T_0^0 \rangle_a = \frac{E_{\mathrm{cyl}}^{\mathrm{out}}(r)}{\pi r^2}, \tag{17.35}$$

where $E_{\mathrm{cyl}}^{\mathrm{out}}(r)$ is the total energy (per unit length) outside the cylindrical surface of radius r. Combining this relation with the similar relation in the interior region, for the total vacuum energy of the cylindrical shell per unit length we obtain

$$E_{\mathrm{cyl}} = E_{\mathrm{cyl}}^{\mathrm{in}}(a) + E_{\mathrm{cyl}}^{\mathrm{out}}(a) = \pi a^2 \left[\langle T_1^1 \rangle_a(a+) - \langle T_1^1 \rangle_a(a-) \right]. \tag{17.36}$$

By taking into account the corresponding expressions for the radial component this yields

$$\begin{aligned} E_{\rm cyl} &= \frac{-1}{4\pi a^2}\sum_{n=0}^{\infty}{}' \int_0^\infty dz\,\chi_\mu(z/a)\big(\ln[I_n(z)K_n(z)]\big)'\left[z^2+\left(z^2+n^2\right)\frac{I_n(z)K_n(z)}{I_n'(z)K_n'(z)}\right] \\ &= \frac{-1}{4\pi a^2}\sum_{n=0}^{\infty}{}' \int_0^\infty dz\,\chi_\mu(z/a)z^2\frac{d}{dz}\ln\left[1-z^2\big(I_n(z)K_n(z)\big)'^2\right]. \end{aligned} \tag{17.37}$$

In the last expression, integrating by part and omitting the boundary term we obtain the Casimir energy in the form used in numerical calculations. The corresponding results are presented in [123, 124, 125]. Note that in the evaluation of the Casimir energy for a perfectly conducting cylindrical shell by the Green function method to perform the complex frequency rotation procedure an additional cutoff function has to be introduced (see [123]). This is related to the above mentioned divergence of the integrals over z for $r=a$. The results of the numerical evaluations for the energy density and pressures distributions (formula (17.33)) are presented in [28, 39]. The energy density, $\langle T_0^0\rangle_a$, and azimuthal pressure, $-\langle T_2^2\rangle_a$, in the exterior region are positive, and the radial pressure, $-\langle T_1^1\rangle_a$, is negative. The ratio of the energy density to the azimuthal pressure is a decreasing function on r, and $1/3 \leqslant -\langle T_0^0\rangle_a/\langle T_2^2\rangle_a \leqslant 0.5$. Note that this ratio is a continuous function for all r and monotonically decreases from 1 at the cylinder axis to 1/3 at infinity.

The quantities (17.27) with (17.29) and (17.33) represent the renormalized VEV of the EMT in the region between two coaxial conducting cylindrical surfaces. Let us consider the limiting cases of the term (17.29). First let $a/r,\ a/b \ll 1$. After replacing $z\to bz$ and expanding the integrand over a/r and a/b it can be seen that

$$\langle T_i^k\rangle_{ab} \approx \langle T_i^k\rangle_b, \quad a/r,\ a/b \ll 1,\ r<b, \tag{17.38}$$

where $\langle T_i^k\rangle_b$ is the vacuum EMT inside a single cylindrical shell with radius b. When $a\to b$ the sum over n in (17.29) diverges. Consequently for $b-a\ll b$ the main contribution comes from large n. By using the uniform asymptotic expansions for the modified Bessel functions, in this limit one obtains the corresponding quantity for the Casimir parallel plate configuration.

From (17.27) and (17.29) it can be seen that the VEV of the EMT can also be written in the form

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle^{(b)} + \langle T_i^k\rangle_{ba}, \quad a<r<b, \tag{17.39}$$

where

$$\langle T_i^k\rangle_{ba} = \frac{\delta_i^k}{4\pi^2}\sum_{n=0}^{\infty}{}' \int_0^\infty dz \sum_{\lambda=0,1} z^3 \frac{F_n^{(i)}[Q_{\lambda n}(bz,rz)]I_n^{(\lambda)}(az)/I_n^{(\lambda)}(bz)}{K_n^{(\lambda)}(az)I_n^{(\lambda)}(bz)-K_n^{(\lambda)}(bz)I_n^{(\lambda)}(az)}. \tag{17.40}$$

The quantities (17.40) are finite for all $a<r\leqslant b$ and diverge on the surface $r=a$.

Now we turn to the interaction forces between the cylindrical surfaces due to the vacuum fluctuations. The vacuum force acting per unit surface at $r=j$ is determined by the $^1_1$–component of the EMT evaluated at this point. The corresponding effective pressures are presented as a sum of the self-action and interaction parts, similar to (17.18). Substituting into formulae (17.29) and (17.40) with $i=k=1$ the values $r=a$ and $r=b$, respectively, and using the Wronskian for the modified Bessel functions, for the interaction parts of the vacuum

pressures on the cylindrical surfaces one has [40]:

$$p_{(\mathrm{int})}^{(a)} = \frac{-1}{4\pi^2 a^4} \sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1} (-1)^\lambda \int_0^\infty dz \, \frac{z\left(n^2/z^2+1\right)^\lambda K_n^{(\lambda)}(bz/a)/K_n^{(\lambda)}(z)}{K_n^{(\lambda)}(z) I_n^{(\lambda)}(bz/a) - K_n^{(\lambda)}(bz/a) I_n^{(\lambda)}(z)}, \quad (17.41)$$

$$p_{(\mathrm{int})}^{(b)} = \frac{-1}{4\pi^2 b^4} \sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1} (-1)^\lambda \int_0^\infty dz \, \frac{z\left(n^2/z^2+1\right)^\lambda I_n^{(\lambda)}(az/b)/I_n^{(\lambda)}(z)}{K_n^{(\lambda)}(az/b) I_n^{(\lambda)}(z) - K_n^{(\lambda)}(z) I_n^{(\lambda)}(az/b)}. \quad (17.42)$$

In (17.41) and (17.42) the summands with $\lambda = 0$ ($\lambda = 1$) come from the electric (magnetic) waves contribution. By using the properties of the modified Bessel functions it can be seen that the quantities $p_{(\mathrm{int})}^{(j)}$ are negative and, hence the corresponding interaction forces are attractive. Note that interaction forces (17.41), (17.42) can also be obtained by differentiating the corresponding part in the vacuum energy with respect to the radii [126].

18 Polarization of the Fulling-Rindler vacuum by a uniformly accelerated mirror

It is well known that the uniqueness of the vacuum state is lost when we work within the framework of quantum field theory in a general curved spacetime or in non–inertial frames. In particular, the use of general coordinate transformations in quantum field theory in flat spacetime leads to an infinite number of unitary inequivalent representations of the commutation relations. Different inequivalent representations will in general give rise to different vacuum states. For instance, the vacuum state for a uniformly accelerated observer, the Fulling-Rindler vacuum, turns out to be inequivalent to that for an inertial observer, the familiar Minkowski vacuum. An interesting topic in the investigations of the Casimir effect is the dependence of the vacuum characteristics on the type of the vacuum. In this section, by using the GAPF, we will study the scalar vacuum polarization brought about by the presence of infinite plane boundary moving by uniform acceleration through the Fulling-Rindler vacuum. The corresponding VEVs of the EMT were studied by Candelas and Deutsch [131] for conformally coupled 4D Dirichlet and Neumann massless scalar and electromagnetic fields. In this paper only the region of the right Rindler wedge to the right of the barrier is considered. In [48] we have investigated the Wightman function, the VEVs of the field square and the EMT for a massive scalar field with general curvature coupling parameter, satisfying Robin boundary condition on infinite plate in an arbitrary number of spacetime dimensions and for the electromagnetic field. The both regions, including the one between the barrier and Rindler horizon are considered.

18.1 Wightman function

In the accelerated reference frame it is convenient to introduce Rindler coordinates $(\tau, \xi, \mathbf{x})$ related to the Minkowski ones, $(t, x^1, \mathbf{x})$ by the formulae $t = \xi \sinh \tau$, $x^1 = \xi \cosh \tau$, and $\mathbf{x} = (x^2, \ldots, x^D)$ denotes the set of coordinates parallel to the plate. In these coordinates the Minkowski line element takes the form

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - d\mathbf{x}^2, \quad (18.1)$$

and a world-line defined by $\xi, \mathbf{x} = \mathrm{const}$ describes an observer with constant proper acceleration ξ^{-1}. The Rindler time coordinate τ is proportional to the proper time along a family of uniformly accelerated trajectories which fill the Rindler wedge, with the proportionality constant equal to the acceleration.

We will assume that the plate is located at $\xi = a$, with a^{-1} being the proper acceleration, and the field satisfies boundary condition (9.11). We will consider the region on the right from the boundary, $\xi \geqslant a$. In Rindler coordinates the boundary condition takes the form

$$(\tilde{A} + \tilde{B}\partial_\xi)\varphi = 0, \quad \xi = a. \tag{18.2}$$

In the region $\xi > a$, a complete set of solutions that are of positive frequency with respect to $\partial/\partial\tau$ and bounded as $\xi \to \infty$ is

$$\varphi_\sigma(x) = \beta_\sigma K_{i\omega}(\lambda\xi)e^{i\mathbf{kx}-i\omega\tau},\ \lambda = \sqrt{k^2+m^2},\ \sigma = (\omega, \mathbf{k}). \tag{18.3}$$

From boundary condition (18.2) we find that the possible values for ω have to be zeros of the function $\bar{K}_{i\omega}(\lambda a)$, where the barred notation is defined by (5.1) with the coefficients

$$A = \tilde{A},\ B = \tilde{B}/a. \tag{18.4}$$

We will denote these zeros by $\omega = \omega_n = \omega_n(k)$, $n = 1, 2, ...$, arranged in ascending order: $\omega_n < \omega_{n+1}$. The coefficient β_σ in (18.3) is determined by the normalization condition with respect to the standard Klein-Gordon inner product, with the ξ-integration over the region (a, ∞). From this condition one finds

$$\beta_\sigma^2 = \frac{1}{(2\pi)^{D-1}} \frac{\bar{I}_{i\omega_n}(\lambda a)}{\partial_\omega \bar{K}_{i\omega}(\lambda a)\mid_{\omega=\omega_n}}. \tag{18.5}$$

Substituting the eigenfunctions (18.3) into (9.8), for the Wightman function we obtain

$$W(x, x') = \int d\mathbf{k}\, \frac{e^{i\mathbf{k}\Delta\mathbf{x}}}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} \frac{\bar{I}_{i\omega}(\lambda a)}{\partial_\omega \bar{K}_{i\omega}(\lambda a)} K_{i\omega}(\lambda\xi)K_{i\omega}(\lambda\xi')e^{-i\omega\Delta\tau}|_{\omega=\omega_n}, \tag{18.6}$$

with $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$, $\Delta\tau = \tau - \tau'$. For the further evaluation of VEV (18.6) we apply to the sum over n summation formula (5.12). As a function $F(z)$ in this formula we choose

$$F(z) = K_{iz}(\lambda\xi)K_{iz}(\lambda\xi')e^{-iz\Delta\tau}. \tag{18.7}$$

Using the asymptotic formulae for the modified Bessel functions it can be seen that condition (5.5) is satisfied if $a^2 e^{|\Delta\tau|} < \xi\xi'$. In particular, this is the case in the coincidence limit $\tau = \tau'$ for the points in the region under consideration, $\xi, \xi' > a$. With $F(z)$ from (18.7), the contribution corresponding to the integral term on the left of formula (5.12) is the Wightman function for the Fulling-Rindler vacuum without boundaries:

$$W_{\mathrm{R}}(x, x') = \frac{1}{\pi^2} \int d\mathbf{k}\, \frac{e^{i\mathbf{k}\Delta\mathbf{x}}}{(2\pi)^{D-1}} \int_0^\infty d\omega \sinh(\pi\omega)e^{-i\omega\Delta\tau}K_{i\omega}(\lambda\xi)K_{i\omega}(\lambda\xi'). \tag{18.8}$$

Taking into account this and applying summation formula (5.12) to Eq. (18.6), we receive [48]

$$W(x, x') = W_{\mathrm{R}}(x, x') + \langle\varphi(x)\varphi(x')\rangle_a, \tag{18.9}$$

where the second term on the right is induced by the barrier:

$$\langle\varphi(x)\varphi(x')\rangle_a = -\frac{1}{\pi} \int d\mathbf{k}\, \frac{e^{i\mathbf{k}\Delta\mathbf{x}}}{(2\pi)^{D-1}} \int_0^\infty d\omega \frac{\bar{I}_\omega(\lambda a)}{\bar{K}_\omega(\lambda a)} K_\omega(\lambda\xi)K_\omega(\lambda\xi')\cosh(\omega\Delta\tau), \tag{18.10}$$

and is finite for $\xi > a$. The divergences in the coincidence limit are contained in the first term corresponding to the Fulling-Rindler vacuum without boundaries. The boundary-induced part in the Wightman function in the region $0 < \xi < a$ is obtained from (18.10) by the replacements $I \rightleftarrows K$ [48] (see below).

18.2 Vacuum expectation values of the field square and the energy-momentum tensor

In the coincidence limit, from formula (18.10) for the boundary part of the field square we have

$$\langle\varphi^2\rangle_a = \frac{-2^{2-D}}{\pi^{(D+1)/2}\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dk\, k^{D-2}\int_0^\infty d\omega\, \frac{\bar{I}_\omega(\lambda a)}{\bar{K}_\omega(\lambda a)}K_\omega^2(\lambda\xi), \quad \xi > a. \tag{18.11}$$

This quantity is monotone increasing negative function on ξ for Dirichlet scalar and monotone decreasing positive function for Neumann scalar. Substituting the function (18.9) into Eq. (9.10) and taking into account Eqs. (18.10) and (18.11), for the VEV of the EMT in the region $\xi > a$ one finds

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle_{\rm R} + \langle T_i^k\rangle_a, \tag{18.12}$$

where the first term on the right is the VEV for the Fulling-Rindler vacuum without boundaries,

$$\langle 0|T_i^k|0\rangle_{\rm R} = \frac{-2^{2-D}\delta_i^k}{\pi^{(D+3)/2}\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dk k^{D-2}\lambda^2\int_0^\infty d\omega e^{-\pi\omega} f^{(i)}\left[K_{i\omega}(\lambda\xi)\right], \tag{18.13}$$

and the second term is the contribution brought by the presence of the barrier:

$$\langle T_i^k\rangle_a = \frac{-2^{2-D}\delta_i^k}{\pi^{(D+1)/2}\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dk k^{D-2}\lambda^2\int_0^\infty d\omega \frac{\bar{I}_\omega(\lambda a)}{\bar{K}_\omega(\lambda a)}F^{(i)}\left[K_\omega(\lambda\xi)\right]. \tag{18.14}$$

In formula (18.13) we have introduced the notations

$$\begin{aligned}
f^{(0)}[g(z)] &= \left(\frac{1}{2}-2\zeta\right)\left|\frac{dg(z)}{dz}\right|^2 + \frac{\zeta}{z}\frac{d}{dz}|g(z)|^2 + \left[\frac{1}{2}-2\zeta+\left(\frac{1}{2}+2\zeta\right)\frac{\omega^2}{z^2}\right]|g(z)|^2, \\
f^{(1)}[g(z)] &= -\frac{1}{2}\left|\frac{dg(z)}{dz}\right|^2 - \frac{\zeta}{z}\frac{d}{dz}|g(z)|^2 + \frac{1}{2}\left(1-\frac{\omega^2}{z^2}\right)|g(z)|^2, \\
f^{(i)}[g(z)] &= \left(\frac{1}{2}-2\zeta\right)\left[\left|\frac{dg(z)}{dz}\right|^2 + \left(1-\frac{\omega^2}{z^2}\right)|g(z)|^2\right] - \frac{\lambda^2-m^2}{(D-1)\lambda^2}|g(z)|^2,
\end{aligned} \tag{18.15}$$

with $i = 2, 3, ..., D$, and the expressions for the functions $F^{(i)}[g(z)]$ are obtained by the replacement $\omega \to i\omega$:

$$F^{(i)}[g(z)] = f^{(i)}[g(z), \omega \to i\omega]. \tag{18.16}$$

The boundary-induced parts in the VEVs of the field square and the EMT for the region $\xi < a$ are obtained from (18.11) and (18.14) with the replacements $I \rightleftarrows K$. It can be easily checked that both summands on the right of Eq. (18.12) satisfy the continuity equation $\nabla_k T_i^k = 0$, which for the geometry under consideration takes the form $\partial_\xi(\xi T_1^1) - T_0^0 = 0$.

The purely Fulling-Rindler part (18.13) of the EMT is investigated in a large number of papers (see, for instance, references given in [49]). The most general case of a massive scalar field in an arbitrary number of spacetime dimensions has been considered in Ref. [132] for conformally and minimally coupled cases and in Ref. [48] for general values of the curvature coupling parameter. For a massless scalar the VEV for the Rindler part without boundaries can be presented in the form

$$\begin{aligned}
\langle T_i^k\rangle_{\rm R,sub} &= \langle 0|T_i^k|0\rangle_{\rm R} - \langle 0|T_i^k|0\rangle_{\rm M} \\
&= -\frac{2\delta_i^k\xi^{-D-1}}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty d\omega \frac{\omega^D g^{(i)}(\omega)}{e^{2\pi\omega}+(-1)^D},
\end{aligned} \tag{18.17}$$

where the expressions for the functions $g^{(i)}(\omega)$ are presented in Ref. [48], and $\langle 0|T_i^k|0\rangle_{\mathrm{M}}$ is the EMT for Minkowski vacuum without boundaries. Expression (18.17) corresponds to the absence from the vacuum of thermal distribution with standard temperature $T = (2\pi\xi)^{-1}$. In general, the corresponding spectrum has non-Planckian form: the density of states factor is not proportional to $\omega^{D-1}d\omega$. The spectrum takes the Planckian form for conformally coupled scalars in $D = 1, 2, 3$ with $g^{(0)}(\omega) = -Dg^{(i)}(\omega) = 1$, $i = 1, 2, \dots D$. It is of interest to note that for even values of spatial dimension the distribution is Fermi-Dirac type (see also [133]). For a massive scalar the energy spectrum is not strictly thermal and the corresponding quantities do not coincide with ones for the Minkowski thermal bath.

Boundary part (18.14) is finite for all values $\xi > a$ and diverges at the plate surface $\xi = a$. To extract the leading part of the boundary divergence, note that near the boundary the main contribution into the ω-integral comes from large values of ω and we can use the uniform asymptotic expansions for the modified Bessel functions [65]. Introducing a new integration variable $k \to \omega k$ and replacing the modified Bessel functions by their uniform asymptotic expansions, it can be seen that the leading terms do not depend on the mass and the Robin coefficients and are the same as for the plate in Minkowski spacetime.

Now let us consider the asymptotic behavior of the boundary part (18.14) for large ξ, $\xi \gg a$. Introducing in Eq. (18.14) a new integration variable $y = \lambda\xi$ and using the asymptotic formulae for the modified Bessel functions for small values of the argument, we see that the integrand is proportional to $(ya/2\xi)^{2\omega}$. It follows from here that the main contribution into the ω-integral comes from small values of ω. Expanding with respect to ω, in the leading order we obtain

$$\langle T_i^k\rangle_a \sim -\frac{\delta_i^k(-1)^{\delta_{A0}}\xi^{-D-1}A_0^{(i)}(m\xi)}{2^D 3\pi^{(D-1)/2}(1+\delta_{A0})\Gamma\left(\frac{D-1}{2}\right)\ln^2(2\xi/a)}, \tag{18.18}$$

where

$$A_0^{(i)}(x) = \int_x^\infty dy\, y^3(y^2 - x^2)^{(D-3)/2}F^{(i)}[K_\omega(y)]|_{\omega=0}. \tag{18.19}$$

If, in addition, one has $m\xi \gg 1$, the integral in this formula can be evaluated replacing the McDonald function by its asymptotic for large values of the argument. In the leading order this yields

$$A_0^{(0)} \approx A_0^{(i)} \approx -2m\xi A_0^{(1)} \approx \pi(1/4 - \zeta)\Gamma\left(\frac{D-1}{2}\right)(m\xi)^{(D+1)/2}e^{-2m\xi}, \tag{18.20}$$

with $i = 2, 3, \cdots, D$. For massless case the integral in Eq. (18.19) may be evaluated and one obtains

$$A_0^{(0)} \approx -DA_1^{(1)} \approx \frac{D}{D-1}A_2^{(2)} \approx \frac{2^D D(\zeta_c - \zeta)}{(D-1)^2\Gamma(D)}\Gamma^4\left(\frac{D+1}{2}\right), \quad m = 0. \tag{18.21}$$

For a conformally coupled scalar the leading terms vanish and the VEVs are proportional to $\xi^{-D-1}\ln^{-3}(2\xi/a)$ for $\xi/a \gg 1$. From Eq. (18.18) we see that for a given ξ the boundary part tends to zero as $a \to 0$ (the barrier coincides with the right Rindler horizon) and the corresponding VEVs of the EMT are the same as for the Fulling-Rindler vacuum without boundaries. Hence, the barrier located at the Rindler horizon does not alter the vacuum EMT.

And finally, we turn to the asymptotic $a, \xi \to \infty$, $\xi - a = \mathrm{const}$. In this limit $\xi/a \to 1$ and ω-integrals in Eq. (18.14) are dominated by large ω. Replacing the modified Bessel functions by their uniform asymptotic expansions, keeping the leading terms only and introducing a new integration variable $\nu = \omega/a$, we can see that the VEVs coincide with the corresponding VEVs induced by a single plate in the Minkowski spacetime with Robin boundary condition on it.

18.3 Electromagnetic field

We now turn to the case of the electromagnetic field in the region $\xi > a$ for the case $D = 3$. We will assume that the mirror is a perfect conductor with the standard boundary conditions of vanishing of the normal component of the magnetic field and the tangential components of the electric field, evaluated at the local inertial frame in which the conductor is instantaneously at rest. As it has been shown in [131], the corresponding eigenfunctions for the vector potential $A_l(x)$ may be resolved into the transverse electric (TE) and transverse magnetic (TM) (with respect to ξ-direction) modes $A_{\alpha\sigma l}$, $\sigma = (\omega, \mathbf{k})$:

$$A_{0\sigma l}(x) = (0, 0, -ik_3, ik_2)\varphi_{0\sigma}(x), \ \alpha = 0, \tag{18.22}$$
$$A_{1\sigma l}(x) = (-\xi\partial_\xi, i\omega/\xi, 0, 0)\,\varphi_{1\sigma}(x), \ \alpha = 1, \tag{18.23}$$

where the eigenfunctions $\varphi_{\sigma\alpha}(x)$ are given by formula (18.3) with $m = 0$, $\mathbf{k} = (k_2, k_3)$, and $\alpha = 0, 1$ correspond to the TE and TM waves, respectively. From the perfect conductor boundary conditions on the vector potential we obtain the corresponding boundary conditions for the scalar modes $\varphi_{\sigma\alpha}(x)$:

$$\varphi_{0\sigma} = 0, \ \partial_\xi\varphi_{1\sigma} = 0, \ \xi = a. \tag{18.24}$$

As a result the TE/TM modes correspond to Dirichlet/Neumann scalars. In the corresponding expressions for the eigenfunctions $A_{\alpha\sigma l}(x)$ the normalization coefficient is determined from the orthonormality condition

$$\int d\mathbf{x} \int_a^\infty \frac{d\xi}{\xi} A^l_{\alpha\sigma} A^*_{\alpha'\sigma' l} = -\frac{2\pi}{\omega}\delta_{\alpha\alpha'}\delta_{\sigma\sigma'}. \tag{18.25}$$

The eigenvalues for ω are the zeros of the function $K_{i\omega}(ka)$, $k = |\mathbf{k}|$, for the TE modes and the zeros of the function $K'_{i\omega}(ka)$ for the TM modes.

Substituting the eigenfunctions for $A_{\alpha\sigma l}(x)$ into mode-sum formula (14.7) with the standard bilinear form for the electromagnetic field EMT and applying to the sums over ω_n formula (5.8), we find

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle_{\rm R} - \frac{\delta_i^k}{4\pi^2}\int_0^\infty dk\, k^3 \int_0^\infty d\omega \left[\frac{I_\omega(ka)}{K_\omega(ka)} + \frac{I'_\omega(ka)}{K'_\omega(ka)}\right] F^{(i)}_{\rm em}\left[K_\omega(k\xi)\right], \tag{18.26}$$

where

$$\langle 0|T_i^k|0\rangle_{\rm R} = \langle 0|T_i^k|0\rangle_{\rm M} - \frac{1}{\pi^2\xi^4}\int_0^\infty d\omega \frac{\omega^3+\omega}{e^{2\pi\omega}-1}\,{\rm diag}(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) \tag{18.27}$$

is the VEV for the Fulling-Rindler vacuum without boundaries [131]. In formula (18.26) the notations

$$F^{(i)}_{\rm em}\left[g(z)\right] = (-1)^i g'^2(z) + (1-(-1)^i\omega^2/z^2)g^2(z), \ i = 0, 1, \tag{18.28}$$
$$F^{(i)}_{\rm em}\left[g(z)\right] = -g^2(z), \ i = 2, 3, \tag{18.29}$$

are introduced. Formula (18.26) is derived in [131] by using the Green function method. Note that the ω-integral in (18.27) is equal to 11/240. The VEV for the EMT of the electromagnetic field in the region $\xi < a$ is obtained from (18.26) by the replacements $I_\omega \rightleftarrows K_\omega$ [48]. By using the properties of the modified Bessel functions it can be seen that

$$\frac{I_\omega(ka)}{K_\omega(ka)} + \frac{I'_\omega(ka)}{K'_\omega(ka)} > 0, \ \frac{K_\omega(ka)}{I_\omega(ka)} + \frac{K'_\omega(ka)}{I'_\omega(ka)} < 0, \tag{18.30}$$

and $F_{\rm em}^{(0)}[K_\omega(z)] > 0$, $F_{\rm em}^{(0)}[I_\omega(z)] > 0$, $F_{\rm em}^{(1)}[K_\omega(z)] < 0$, $F_{\rm em}^{(1)}[I_\omega(z)] > 0$. In particular, from these inequalities it follows that the energy density induced by the plate is positive in the region $\xi < a$ and negative in the region $\xi > a$. The corresponding graphs are given in [48, 131]. The behavior of the plate-induced VEVs in various asymptotic regions of the parameters can be found in [48, 131]. In Ref. [134] the formulae derived in this section were used to generate the vacuum densities for a conformally coupled massless scalar field in de Sitter spacetime in presence of a curved brane on which the field obeys the Robin boundary conditions with coordinate dependent coefficients.

19 Vacuum densities in the region between two plates uniformly accelerated through the Fulling-Rindler vacuum

19.1 Wightman function

Consider two parallel plates moving with uniform proper accelerations assuming that the quantum scalar field is prepared in the Fulling-Rindler vacuum [49, 50]. We will let the surfaces $\xi = a$ and $\xi = b$, $b > a > 0$, represent the trajectories of the plates, which therefore have proper accelerations a^{-1} and b^{-1}. We consider the case of a scalar field satisfying Robin boundary conditions on the surfaces of the plates:

$$\left(\tilde{A}_j + \tilde{B}_j \partial_\xi\right) \varphi\Big|_{\xi=j} = 0, \quad j = a, b, \tag{19.1}$$

with constant coefficients $\tilde{A}_j$ and $\tilde{B}_j$. The plates divide the right Rindler wedge into three regions: $0 < \xi < a$, $\xi > b$, and $a < \xi < b$. The VEVs in two first regions are the same as those induced by single plates located at $\xi = a$ and $\xi = b$, respectively. In the region between the plates the eigenfunctions satisfying boundary condition on the plate $\xi = b$ have the form

$$\varphi_\sigma(x) = \beta_\sigma G_{i\omega}^{(b)}(\lambda b, \lambda \xi) e^{i\mathbf{kx} - i\omega\tau}, \tag{19.2}$$

with the function $G_\nu^{(j)}(x, y)$ defined by formula (12.30). Note that the function $G_{i\omega}^{(b)}(\lambda b, \lambda \xi)$ is real, $G_{-i\omega}^{(b)}(\lambda b, \lambda \xi) = G_{i\omega}^{(b)}(\lambda b, \lambda \xi)$. From the boundary condition on the plate $\xi = a$ we find that the possible values for ω are roots to the equation

$$Z_{i\omega}(\lambda a, \lambda b) = 0, \tag{19.3}$$

where the function $Z_{i\omega}(u, v)$ is defined by formula (5.14), and the barred notations are defined by formula (4.2) with $A_j = \tilde{A}_j$, $B_j = \tilde{B}_j/j$. For a fixed λ, the equation (19.3) has an infinite set of real solutions with respect to ω. We will denote them by $\Omega_n = \Omega_n(\lambda a, \lambda b)$, $\Omega_n > 0$, $n = 1, 2, \ldots$, and will assume that they are arranged in ascending order $\Omega_n < \Omega_{n+1}$. In the consideration below we will assume the values of the coefficients in Eq. (19.1) for which the imaginary solutions are absent and the vacuum is stable.

The coefficient β_σ in formula (19.2) is determined from the orthonormality condition. Taking into account the boundary conditions, for this coefficient one finds

$$\beta_\sigma^2 = \left. \frac{(2\pi)^{1-D}\, \bar{I}_{i\omega}^{(a)}(\lambda a)}{\bar{I}_{i\omega}^{(b)}(\lambda b) \partial_\omega Z_{i\omega}(\lambda a, \lambda b)} \right|_{\omega=\Omega_n}. \tag{19.4}$$

Now, substituting the eigenfunctions into mode-sum formula (9.8), for the positive frequency Wightman function one finds

$$
\begin{aligned}
W(x,x') &= \int \frac{d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} \frac{\bar{I}_{i\omega}^{(a)}(\lambda a) e^{-i\omega\Delta\tau}}{\bar{I}_{i\omega}^{(b)}(\lambda b)\partial_\omega Z_{i\omega}(\lambda a,\lambda b)} \\
&\quad \times G_{i\omega}^{(b)}(\lambda b,\lambda\xi) G_{i\omega}^{(b)}(\lambda b,\lambda\xi')\Big|_{\omega=\Omega_n}. \qquad (19.5)
\end{aligned}
$$

For the further evaluation of this VEV we apply to the sum over n summation formula (5.18). As a function $F(z)$ in this formula we choose

$$
F(z) = \frac{G_{i\omega}^{(b)}(\lambda b,\lambda\xi) G_{i\omega}^{(b)}(\lambda b,\lambda\xi')}{\bar{I}_{iz}^{(b)}(\lambda b)\bar{I}_{-iz}^{(b)}(\lambda b)} e^{-iz\Delta\tau}. \qquad (19.6)
$$

Condition (5.19) for this function is satisfied if $a^2 e^{|\Delta\tau|} < \xi\xi'$. By using formula (5.18), for the Wightman function one obtains the expression [50]

$$
\begin{aligned}
W(x,x') &= W^{(b)}(x,x') - \int \frac{d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}}{\pi(2\pi)^{D-1}} \int_0^\infty d\omega\, \Omega_{b\omega}(\lambda a,\lambda b) \\
&\quad \times G_{\omega}^{(b)}(\lambda b,\lambda\xi) G_{\omega}^{(b)}(\lambda b,\lambda\xi') \cosh(\omega\Delta\tau), \qquad (19.7)
\end{aligned}
$$

where we have used the notation (12.35). In Eq. (19.7),

$$
\begin{aligned}
W^{(b)}(x,x') &= \int \frac{d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}}{\pi^2(2\pi)^{D-1}} \int_0^\infty d\omega \sinh(\pi\omega) \\
&\quad \times e^{-i\omega\Delta\tau} \frac{G_{i\omega}^{(b)}(\lambda b,\lambda\xi) G_{i\omega}^{(b)}(\lambda b,\lambda\xi')}{\bar{I}_{i\omega}^{(b)}(\lambda b)\bar{I}_{-i\omega}^{(b)}(\lambda b)}, \qquad (19.8)
\end{aligned}
$$

is the Wightman function in the region $\xi < b$ for a single plate at $\xi = b$. This function is investigated in Ref. [48] and can be presented in the form

$$
W^{(b)}(x,x') = W_{\mathrm{R}}(x,x') + \langle\varphi(x)\varphi(x')\rangle_b, \qquad (19.9)
$$

where $W_{\mathrm{R}}(x,x')$ is the Wightman function for the right Rindler wedge without boundaries and the part

$$
\begin{aligned}
\langle\varphi(x)\varphi(x')\rangle_b &= -\int \frac{d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}}{\pi(2\pi)^{D-1}} \int_0^\infty d\omega \frac{\bar{K}_{\omega}^{(b)}(\lambda b)}{\bar{I}_{\omega}^{(b)}(\lambda b)} \\
&\quad \times I_\omega(\lambda\xi) I_\omega(\lambda\xi') \cosh(\omega\Delta\tau) \qquad (19.10)
\end{aligned}
$$

is induced in the region $\xi < b$ by the presence of the plate at $\xi = b$. Note that the representation (19.9) with (19.10) is valid under the assumption $\xi\xi' < b^2 e^{|\Delta\tau|}$.

By using the identity

$$
\begin{aligned}
&\frac{\bar{K}_{\omega}^{(b)}(\lambda b)}{\bar{I}_{\omega}^{(b)}(\lambda b)} I_\omega(\lambda\xi) I_\omega(\lambda\xi') = \frac{\bar{I}_{\omega}^{(a)}(\lambda a)}{\bar{K}_{\omega}^{(a)}(\lambda a)} K_\omega(\lambda\xi) K_\omega(\lambda\xi') \\
&+ \sum_{j=a,b} n^{(j)} \Omega_{j\omega}(\lambda a,\lambda b) G_{\omega}^{(j)}(\lambda j,\lambda\xi) Z_{\omega}^{(j)}(\lambda j,\lambda\xi'), \qquad (19.11)
\end{aligned}
$$

with $n^{(a)} = 1$, $n^{(b)} = -1$, the Wightman function can also be presented in the form

$$\begin{aligned} W(x,x') &= W^{(a)}(x,x') - \int \frac{d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}}{\pi(2\pi)^{D-1}} \int_0^\infty d\omega\, \Omega_{a\omega}(\lambda a, \lambda b) \\ &\quad \times G_\omega^{(a)}(\lambda a, \lambda\xi) G_\omega^{(a)}(\lambda a, \lambda\xi') \cosh(\omega\Delta\tau). \end{aligned} \tag{19.12}$$

In this formula $W^{(a)}(x,x')$ is the Wightman function in the region $\xi > a$ for a single plate at $\xi = a$, and is investigated in the previous section. In the coincidence limit the second term on the right of formula (19.7) is finite on the plate $\xi = b$ and diverges on the plate at $\xi = a$, whereas the second term on the right of Eq. (19.12) is finite on the plate $\xi = a$ and is divergent for $\xi = b$. Consequently, the form (19.7) ((19.12)) is convenient for the investigations of the VEVs near the plate $\xi = b$ ($\xi = a$).

19.2 Scalar Casimir densities

Here we will consider the VEVs of the field square and the EMT in the region between the plates [50]. In the coincidence limit from the formulae for the Wightman function one obtains two equivalent forms for the VEV of the field square,

$$\begin{aligned} \langle 0 | \varphi^2 | 0 \rangle &= \langle 0 | \varphi^2 | 0 \rangle_{\mathrm{R}} + \langle \varphi^2 \rangle_j \\ &\quad - B_D \int_0^\infty dk\, k^{D-2} \int_0^\infty d\omega\, \Omega_{j\omega}(\lambda a, \lambda b) G_\omega^{(j)2}(\lambda j, \lambda\xi), \end{aligned} \tag{19.13}$$

corresponding to $j = a$ and $j = b$, and B_D is defined by (17.14). In Eq. (19.13) the part $\langle \varphi^2 \rangle_j$ is induced by a single plate at $\xi = j$ when the second plate is absent. The last term on the right of formula (19.13) is finite on the plate at $\xi = j$ and diverges for points on the other plate. Extracting the contribution from this plate, we can write the expression (19.13) for the VEV in the symmetric form

$$\langle 0 | \varphi^2 | 0 \rangle = \langle 0 | \varphi^2 | 0 \rangle_{\mathrm{R}} + \sum_{j=a,b} \langle \varphi^2 \rangle_j + \Delta \langle \varphi^2 \rangle, \tag{19.14}$$

with the interference part

$$\begin{aligned} \Delta \langle \varphi^2 \rangle &= -B_D \int_0^\infty dk\, k^{D-2} \int_0^\infty d\omega \bar{I}_\omega^{(a)}(\lambda a) \\ &\quad \times \left[\frac{G_\omega^{(b)2}(\lambda b, \lambda\xi)}{\bar{I}_\omega^{(b)}(\lambda b) G_\omega^{ab}(\lambda a, \lambda b)} - \frac{K_\omega^2(\lambda\xi)}{\bar{K}_\omega^{(a)}(\lambda a)} \right]. \end{aligned} \tag{19.15}$$

An equivalent form for this part is obtained with the replacements $a \rightleftarrows b$ and $I \rightleftarrows K$ in the integrand. Interference term (19.15) is finite for all values of ξ in the range $a \leqslant \xi \leqslant b$, including the points on the boundaries. The surface divergences are contained in the single plate parts only. In the limit $a \to b$ with fixed values of the coefficients in the boundary conditions, the interference part (19.15) is divergent and for small values of $b/a - 1$ the main contribution comes from large values of ω. Introducing an integration variable $x = k/\omega$ and replacing the modified Bessel functions by their uniform asymptotic expansions, we can see that to the leading order $\Delta\langle\varphi^2\rangle$ coincides with the VEV of the field square in the region between two parallel plates in the Minkowski bulk.

Large values of the proper accelerations for the plates correspond to the limit $a, b \to 0$. In this limit the plates are close to the Rindler horizon. From formulae (18.11), (19.15) we see that

for fixed values of the ratios a/b, ξ/b, both single plate and interference parts behave as b^{1-D} in the limit $b \to 0$. In the limit $a \to 0$ for fixed values ξ and b, the left plate tends to the Rindler horizon for a fixed world-line of the right plate. The main contribution into the ω-integral in Eq. (19.15) comes from small values ω, $\omega \lesssim 1/\ln(2/\lambda a)$. Using the formulae for the modified Bessel functions for small arguments, it can be seen that interference part (19.15) vanishes as $\ln^{-2}(2b/a)$.

In the limit of small accelerations of the plates, $a, b \to \infty$, with fixed values $b-a$, B_j/A_j, and m, the main contribution comes from large values ω. Using the uniform asymptotic formulae for the modified Bessel functions, it can be seen that $\langle\varphi^2\rangle_j$ and $\Delta\langle\varphi^2\rangle$ coincide with the corresponding expressions for the geometry of two parallel plates on the Minkowski bulk. In this limit, ξ corresponds to the Cartesian coordinate perpendicular to the plates which are located at $\xi = a$ and $\xi = b$.

For large values of the mass, $ma \gg 1$, we introduce a new integration variable $y = \lambda/m$. The main contribution into the ω-integral comes from the values $\omega \sim \sqrt{ma}$. By using the uniform asymptotic expansions for the modified Bessel functions for large values of the order and further expanding over ω/ma, for the single plate parts to the leading order one finds

$$\langle\varphi^2\rangle_j \approx -\frac{m^{\frac{D}{2}-1} e^{-2m|\xi-j|}\sqrt{j/\xi}}{2(4\pi)^{\frac{D}{2}} c_j(m)|\xi-j|^{\frac{D}{2}}},\ j = a, b, \tag{19.16}$$

where we have introduced notations

$$c_j(y) = \frac{A_j - n^{(j)}\tilde{B}_j y}{A_j + n^{(j)}\tilde{B}_j y}. \tag{19.17}$$

In the similar way, for the interference part we obtain the formula

$$\Delta\langle\varphi^2\rangle \approx \frac{m^{\frac{D}{2}-1} e^{2m(a-b)}\sqrt{ab}}{(4\pi)^{\frac{D}{2}} \xi c_a(m) c_b(m)(b-a)^{\frac{D}{2}}}. \tag{19.18}$$

As we could expect, the both single plate and interference parts are exponentially suppressed for large values of the mass.

Making use of the formulae for the Wightman function and the field square, one obtains two equivalent forms for the VEV of the EMT, corresponding to $j = a$ and $j = b$ (no summation over i) [50] (see also [49] for the case of Dirichlet and Neumann scalars):

$$\begin{aligned}\langle 0|T_i^k|0\rangle &= \langle 0|T_i^k|0\rangle_{\mathrm{R}} + \langle T_i^k\rangle_j - B_D\delta_i^k \int dk\, k^{D-2} \\ &\times \lambda^2 \int_0^\infty d\omega\, \Omega_{j\omega}(\lambda a, \lambda b) F^{(i)}\left[G_\omega^{(j)}(\lambda j, \lambda\xi)\right]. \end{aligned} \tag{19.19}$$

In this formula, $\langle 0|T_i^k|0\rangle_{\mathrm{R}}$ is the corresponding VEV for the Fulling-Rindler vacuum without boundaries, and the term $\langle T_i^k\rangle_j$ is induced by the presence of a single plane boundary located at $\xi = j$ in the region $\xi > a$ for the case $j = a$ and in the region $\xi < b$ for $j = b$. In formulae (19.19), for a given function $g(z)$ we use the notations $F^{(i)}[g(z)]$ from (18.16). For the last term on the right of Eq. (19.19) we have to substitute $g(z) = G_\omega^{(j)}(\lambda j, z)$. It can be easily seen that for a conformally coupled massless scalar the EMT is traceless.

Now let us present the VEV (19.19) in the form

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle_{\mathrm{R}} + \sum_{j=a,b} \langle T_i^k\rangle_j + \Delta\langle T_i^k\rangle, \tag{19.20}$$

where (no summation over i)

$$\begin{aligned}\Delta\langle T_i^k\rangle &= -A_D\delta_i^k\int_0^\infty dk\,k^{D-2}\lambda^2\int_0^\infty d\omega\bar{I}_\omega^{(a)}(\lambda a)\\ &\times\left[\frac{F^{(i)}[G_\omega^{(b)}(\lambda b,\lambda\xi)]}{\bar{I}_\omega^{(b)}(\lambda b)G_\omega^{ab}(\lambda a,\lambda b)}-\frac{F^{(i)}[K_\omega(\lambda\xi)]}{\bar{K}_\omega^{(a)}(\lambda a)}\right]\end{aligned} \tag{19.21}$$

is the interference term. The surface divergences are contained in the single boundary parts and this term is finite for all values $a\leqslant\xi\leqslant b$. An equivalent formula for $\Delta\langle T_i^k\rangle$ is obtained from Eq. (19.21) by replacements $a\rightleftarrows b$, $I\rightleftarrows K$. Both single plate and interference parts separately satisfy the continuity equation. For a conformally coupled massless scalar field they are traceless and we have an additional relation $\langle T_i^i\rangle=0$.

In the limit $a\to b$ expression (19.21) is divergent and for small values of $b/a-1$ the main contribution comes from large values of ω. Introducing a new integration variable $x=k/\omega$ and replacing the modified Bessel functions by their uniform asymptotic expansions for large values of the order, at the leading order one receives

$$\begin{aligned}\Delta\langle T_i^i\rangle &\approx -\frac{(4\pi)^{-\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}\int_0^\infty dy\frac{y^D}{k_ak_be^{2y(b-a)}-1}\\ &\times\left[1+2D\left(1-\delta_1^i\right)(\zeta-\zeta_D)\sum_{j=a,b}k_je^{-2y|\xi-j|}\right],\end{aligned} \tag{19.22}$$

where $k_j=1-2\delta_{0B_j}$. In the limit of large proper accelerations for the plates, $a,b\to 0$, for fixed values a/b and ξ/b, the world-lines of both plates are close to the Rindler horizon. In this case the single plate and interference parts grow as b^{-D-1}. The situation is essentially different when the world-line of the left plate tends to the Rindler horizon, $a\to 0$, whereas b and ξ are fixed. In the way similar to that for the case of the field square, it can be seen that in this limit the interference part (19.21) vanishes as $\ln^{-2}(2b/a)$.

In the limit of small proper accelerations, $a,b\to\infty$ with fixed values $b-a$, B_j/A_j, and m, the main contribution comes from large values of ω. Using the asymptotic formulae for the modified Bessel functions, to the leading order one obtains the corresponding expressions for the geometry of two parallel plates on the Minkowski background investigated in [32]. In particular, in this limit the single boundary terms vanish for a conformally coupled massless scalar.

For large values of the mass, $ma\gg 1$, by the method similar to that used in the previous subsection for the field square, it can be seen that the both single plate and interference parts are exponentially suppressed (no summation over i): $\langle T_i^i\rangle_j\sim m^{D/2+1}\exp[-2m|\xi-j|]$, $j=a,b$, for single plate parts and $\Delta\langle T_i^i\rangle\sim m^{D/2+1}\exp[2m(a-b)]$ for the interference part.

Now we turn to the interaction forces between the plates due to the vacuum fluctuations. The vacuum force acting per unit surface of the plate at $\xi=j$ is determined by the $^1_1$–component of the vacuum EMT evaluated at this point. The corresponding effective pressures can be presented as a sum of self-action and interactions terms, $p^{(j)}=p_1^{(j)}+p_{(\text{int})}^{(j)}$, $j=a,b$. The first term on the right is the pressure for a single plate at $\xi=j$ when the second plate is absent. The first term on the right is the pressure for a single plate at $\xi=j$ when the second plate is absent. This term is divergent due to the surface divergences in the subtracted VEVs and needs additional renormalization. This can be done, for example, by applying the generalized zeta function technique to the corresponding mode-sum. This procedure is similar to that used in [135] for the evaluation of the total Casimir energy in the cases of Dirichlet and Neumann boundary conditions and in [136] for the evaluation of the surface energy for a single Robin

plate. The term $p_{\rm (int)}^{(j)}$ is the pressure induced by the presence of the second plate. This term is finite for all nonzero distances between the plates and is not affected by the renormalization procedure. For the plate at $\xi = j$ the interaction term is due to the third summand on the right of Eq. (19.19). Substituting into this term $\xi = j$ and using the Wronskian for the modified Bessel functions one finds [50]

$$p_{\rm (int)}^{(j)} = \frac{A_D A_j^2}{2j^2} \int_0^\infty dk\, k^{D-2} \int_0^\infty d\omega \left[\left(\lambda^2 j^2 + \omega^2 \right) \beta_j^2 + 4\zeta\beta_j - 1 \right] \Omega_{j\omega}(\lambda a, \lambda b), \tag{19.23}$$

with $\beta_j = B_j/(jA_j)$. In dependence of the values for the coefficients in the boundary conditions, the effective pressures (19.23) can be either positive or negative, leading to repulsive or attractive forces. It can be seen that for Dirichlet boundary condition on one plate and Neumann boundary condition on the other one has $p_{\rm (int)}^{(j)} > 0$ and the interaction forces are repulsive for all distances between the plates. Note that for Dirichlet or Neumann boundary conditions on both plates the interaction forces are always attractive [49]. The interaction forces can also be written in another equivalent form

$$\begin{aligned} p_{\rm (int)}^{(j)} &= n^{(j)} \frac{A_D}{2j} \int_0^\infty dk\, k^{D-2} \int_0^\infty d\omega \left[1 + \frac{(4\zeta - 1)\,\beta_j}{(\lambda^2 j^2 + \omega^2)\,\beta_j^2 + \beta_j - 1} \right] \\ &\quad \times \frac{\partial}{\partial j} \ln \left| 1 - \frac{\bar{I}_\omega^{(a)}(\lambda a) \bar{K}_\omega^{(b)}(\lambda b)}{\bar{I}_\omega^{(b)}(\lambda b) \bar{K}_\omega^{(a)}(\lambda a)} \right|. \end{aligned} \tag{19.24}$$

For Dirichlet and Neumann scalars the second term in the square brackets is zero.

The results obtained above can be applied to the geometry of two parallel plates near the $D = 3$ 'Rindler wall.' This wall is described by the static plane-symmetric distribution of the matter with the diagonal EMT $T_i^k = \mathrm{diag}(\varepsilon_m, -p_m, -p_m, -p_m)$ (see Ref. [137]). Below we will denote by x the coordinate perpendicular to the wall and will assume that the plane $x = 0$ is at the center of the wall. If the plane $x = x_s$ is the boundary of the wall, when the external $(x > x_s)$ line element with the time coordinate t can be transformed into form (18.1) with

$$\xi(x) = x - x_s + \frac{1}{2\pi\sigma_s}, \quad \tau = 2\pi\sigma_s \sqrt{g_{00}(x_s)}\, t. \tag{19.25}$$

In this formula the parameter σ_s is the mass per unit surface of the wall and is determined by the distribution of the matter:

$$\sigma_s = 2 \int_0^{x_s} (\varepsilon_m + 3p_m) \left[g(x)/g(x_s) \right]^{1/2} dx. \tag{19.26}$$

For the 'Rindler wall' one has $g_{22}'(x)|_{x=0} < 0$ [137] (the external solution for the case $g_{22}'(x)|_{x=0} > 0$ is described by the standard Taub metric). Hence, the Wightman function, the VEVs for the field square and the EMT in the region between two plates located at $x = x_a$ and $x = x_b$, $x_j > x_s$ near the 'Rindler wall' are obtained from the results given above substituting $j = \xi(x_j)$, $j = a, b$ and $\xi = \xi(x)$. For $\sigma_s > 0$, $x \geqslant x_s$ one has $\xi(x) \geqslant \xi(x_s) > 0$ and the Rindler metric is regular everywhere in the external region.

19.3 Electromagnetic field

As in the case of a single plate considered in subsection 18.3, in the region between two uniformly accelerated perfectly conducting plates the eigenfunctions for the electromagnetic vector

potential are resolved int TE and TM modes in accordance with (18.22), (18.23), with Dirichlet and Neumann boundary conditions:

$$\varphi_{0\sigma}|_{\xi=a} = \varphi_{0\sigma}|_{\xi=b} = 0, \quad \partial_\xi \varphi_{1\sigma}|_{\xi=a} = \partial_\xi \varphi_{1\sigma}|_{\xi=b} = 0. \tag{19.27}$$

The corresponding eigenvalues for ω are the zeros of the function

$$Z_{\alpha,i\omega}(ka,kb) = I_{i\omega}^{(\alpha)}(kb)K_{i\omega}^{(\alpha)}(ka) - K_{i\omega}^{(\alpha)}(kb)I_{i\omega}^{(\alpha)}(ka), \tag{19.28}$$

where $\alpha = 0,1$ for the TE and TM modes respectively, and $f_{i\omega}^{(0)}(x) = f_{i\omega}(x)$, $f_{i\omega}^{(1)}(x) = f'_{i\omega}(x)$ for $f = I, K$. The eigenfunctions for the separate scalar modes $\varphi_{\alpha\sigma}(x)$ are given by formula (19.2) with $m = 0$, $D = 3$ (for the case of an arbitrary D see [135]) and with the replacement $G_{i\omega}^{(b)}(\lambda b, \lambda\xi) \to G_{\alpha,i\omega}(k\xi, kb)$, $\alpha = 0,1$, where

$$G_{\alpha,\omega}(x,y) = I_\omega^{(\alpha)}(y)K_\omega(x) - K_\omega^{(\alpha)}(y)I_\omega(x). \tag{19.29}$$

The corresponding normalization coefficients are determined from (18.25) where now the ξ-integration goes over the interval (a,b).

Substituting the normalized eigenfunctions into the mode-sum formula for the VEV of the EMT and applying formula (5.18) for the summation over the eigenvalues of ω, similar to the case of a scalar field one finds [49]

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle^{(b)} - \frac{\delta_i^k}{4\pi^2}\int_0^\infty dk\, k^3 \int_0^\infty d\omega \sum_{\alpha=0,1} \frac{I_\omega^{(\alpha)}(ka)F_{\rm em}^{(i)}[G_{\alpha,\omega}(k\xi,kb)]}{I_\omega^{(\alpha)}(kb)Z_{\alpha,\omega}(ka,kb)}, \tag{19.30}$$

where $\langle T_i^k\rangle^{(b)}$ is the VEV in the region $\xi < b$ corresponding to the geometry of a single plate with $\xi = b$ and the functions $F_{\rm em}^{(i)}[g(z)]$ are defined by formulae (18.28), (18.29) with $g(z) = G_{\alpha,\omega}(z,kb)$. An alternative form for the vacuum EMT in the region between two plates is

$$\langle 0|T_i^k|0\rangle = \langle T_i^k\rangle^{(a)} - \frac{\delta_i^k}{4\pi^2}\int_0^\infty dk\, k^3 \int_0^\infty d\omega \sum_{\alpha=0,1} \frac{K_\omega^{(\alpha)}(kb)F_{\rm em}^{(i)}[G_{\alpha,\omega}(k\xi,ka)]}{K_\omega^{(\alpha)}(ka)Z_{\alpha,\omega}(ka,kb)}, \tag{19.31}$$

where $\langle T_i^k\rangle^{(a)}$ is the vacuum EMT in the region $\xi > a$ corresponding to the geometry of a single boundary at $\xi = a$.

For the interaction force $p_{\rm em(int)}^{(j)}$, $j = a,b$, per unit area of the plate at $\xi = j$, from Eqs. (19.30) and (19.31) one obtains [49]

$$\begin{aligned} p_{\rm em(int)}^{(a)} &= -\frac{1}{4\pi^2 a^2}\int_0^\infty dk\, k \int_0^\infty d\omega \sum_{\alpha=0,1} (-1)^\alpha \frac{K_\omega^{(\alpha)}(kb)}{K_\omega^{(\alpha)}(ka)} \frac{(1+\omega^2/k^2a^2)^\alpha}{Z_{\alpha,\omega}(ka,kb)}, && (19.32)\\ p_{\rm em(int)}^{(b)} &= -\frac{1}{4\pi^2 b^2}\int_0^\infty dk\, k \int_0^\infty d\omega \sum_{\alpha=0,1} (-1)^\alpha \frac{I_\omega^{(\alpha)}(ka)}{I_\omega^{(\alpha)}(kb)} \frac{(1+\omega^2/k^2b^2)^\alpha}{Z_{\alpha,\omega}(ka,kb)}. && (19.33)\end{aligned}$$

Recalling that $(-1)^\alpha Z_{\alpha,\omega}(ka,kb) > 0$ we see that the electromagnetic interaction forces are attractive. Note that $p_{\rm em(int)}^{(j)} = p_{D({\rm int})}^{(j)} + p_{N({\rm int})}^{(j)}$, where $p_{D({\rm int})}^{(j)}$ and $p_{N({\rm int})}^{(j)}$ are the interaction forces for Dirichlet and Neumann scalars. In the limit $a \to b$, to the leading order of $1/(b-a)$ from these expressions the electromagnetic Casimir interaction force between two parallel plates in the Minkowski spacetime is obtained. Note that the interaction forces (19.32), (19.33) can also be obtained by the differentiation the corresponding Casimir energy [135].

20 Wightman function and Casimir densities for branes on AdS bulk

Anti-de Sitter (AdS) spacetime is one of the simplest and most interesting spacetimes allowed by general relativity. Quantum field theory in this background has been discussed by several authors. Much of early interest to AdS spacetime was motivated by the questions of principle nature related to the quantization of fields propagating on curved backgrounds. The importance of this theoretical work increased when it was realized that AdS spacetime emerges as a stable ground state solution in extended supergravity and Kaluza-Klein models and in string theories. The appearance of the AdS/CFT correspondence and braneworld models of Randall-Sundrum type [138] has revived interest in this subject considerably. Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention (see, for instance, [57] for relevant references). In this section we apply the GAPF for the investigation of the positive frequency Wightman function and VEV of the EMT for a massive scalar field with general curvature coupling parameter subject to Robin boundary conditions on two parallel branes located on $(D+1)$-dimensional AdS background [56]. The general case of different Robin coefficients on separate branes is considered.

20.1 Wightman function

Consider a scalar field $\varphi(x)$ on background of a $(D+1)$-dimensional plane-symmetric spacetime with the line element

$$ds^2 = g_{ik}dx^i dx^k = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{20.1}$$

and with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$ being the metric for the D-dimensional Minkowski spacetime. Here and below in this section $i,k = 0,1,\ldots,D$, and $\mu,\nu = 0,1,\ldots,D-1$. By making a coordinate transformation

$$z = \int e^{\sigma(y)}dy, \tag{20.2}$$

metric (20.1) is written in a conformally-flat form $ds^2 = e^{-2\sigma(y)}\eta_{ik}dx^i dx^k$.

We will study quantum vacuum effects brought about by the presence of parallel infinite plane boundaries (branes), located at $y = a$ and $y = b$, $a < b$, with mixed boundary conditions

$$\left(\tilde{A}_y + \tilde{B}_y\partial_y\right)\varphi(x) = 0, \quad y = a, b, \tag{20.3}$$

and constant coefficients $\tilde{A}_y$, $\tilde{B}_y$. As a first stage we will consider the positive frequency Wightman function. To apply the mode-sum formula we need the eigenfunctions for the problem under consideration. On the base of the plane symmetry of the problem these functions can be presented in the form

$$\varphi_\alpha(x^i) = \frac{e^{-i\eta_{\mu\nu}k^\mu x^\nu}}{\sqrt{2\omega(2\pi)^{D-1}}})f_n(y),\ k^\mu = (\omega,\mathbf{k}), \tag{20.4}$$

where $\omega = \sqrt{k^2+m_n^2}$, $k = |\mathbf{k}|$, and the separation constants m_n are determined by the boundary conditions. Substituting eigenfunctions (20.4) into the field equation, for the function $f_n(y)$ one obtains the following equation

$$-e^{D\sigma}\partial_y\left(e^{-D\sigma}\partial_y f_n\right) + \left(m^2+\zeta R\right)f_n = m_n^2 e^{2\sigma}f_n. \tag{20.5}$$

For the AdS geometry one has $\sigma(y) = k_D y$, $z = e^{\sigma(y)}/k_D$, and $R = -D(D+1)k_D^2$, where the AdS curvature radius is given by $1/k_D$. In this case the solution to equation (20.5) for the region $a < y < b$ satisfying the boundary condition at $y = b$ is given by

$$f_n(y) = c_n e^{D\sigma/2} g_\nu(m_n z_a, m_n z), \tag{20.6}$$

where the function $g_\nu(u,v)$ is defined by formula (17.3),

$$\nu = \sqrt{(D/2)^2 - D(D+1)\zeta + m^2/k_D^2}, \quad z_j = e^{\sigma(j)}/k_D, \tag{20.7}$$

and we use the barred notation (4.2) with

$$A_j = \tilde{A}_j + \tilde{B}_j k_D D/2, \quad B_j = \tilde{B}_j k_D, \quad j = a, b. \tag{20.8}$$

We will assume values of the curvature coupling parameter for which ν is real. For imaginary ν the ground state becomes unstable [139]. Note that for a conformally coupled massless scalar one has $\nu = 1/2$ and the cylinder functions in Eq. (20.6) are expressed via the elementary functions. From the boundary condition on the brane $y = b$ we receive that the eigenvalues m_n have to be solutions to the equation

$$C_\nu^{ab}(z_b/z_a, m_n z_a) \equiv \bar{J}_\nu^{(a)}(m_n z_a)\bar{Y}_\nu^{(b)}(m_n z_b) - \bar{Y}_\nu^{(a)}(m_n z_a)\bar{J}_\nu^{(b)}(m_n z_b) = 0. \tag{20.9}$$

The eigenvalues for m_n are related to the zeros of the function $C_\nu^{ab}(\eta, z)$ as

$$m_n = k_D \gamma_{\nu,n} e^{-\sigma(a)} = \gamma_{\nu,n}/z_a. \tag{20.10}$$

The coefficient c_n in Eq. (20.6) is determined from the orthonormality condition

$$\int_a^b dy e^{(2-D)\sigma} f_n(y) f_{n'}(y) = \delta_{nn'}, \tag{20.11}$$

and is equal to

$$c_n^2 = \frac{\pi^2 u}{2k_D z_a^2} T_\nu^{ab}(\eta, u), \quad u = \gamma_{\nu,n}, \quad \eta = z_b/z_a. \tag{20.12}$$

Note that, as we consider the quantization in the region between the branes, $z_a \leqslant z \leqslant z_b$, the modes defined by (20.6) are normalizable for all real values of ν from Eq. (20.7).

Substituting the eigenfunctions (20.4) into mode-sum (9.8), for the expectation value of the field product one finds

$$W(x,x') = \frac{k_D^{D-1}(zz')^{D/2}}{2^{D+1}\pi^{D-3} z_a^2} \int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}} \sum_{n=1}^{\infty} h_\nu(\gamma_{\nu,n}) T_\nu^{ab}(\eta, \gamma_{\nu,n}), \tag{20.13}$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^{D-1})$ represents the coordinates in $(D-1)$-hyperplane parallel to the branes, $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$, and

$$h_\nu(u) = \frac{u e^{-i\Delta t\sqrt{u^2/z_a^2 + k^2}}}{\sqrt{u^2/z_a^2 + k^2}} g_\nu(u, uz/z_a) g_\nu(u, uz'/z_a), \tag{20.14}$$

with $\Delta t = t - t'$.

To sum over n we will use summation formula (4.14). Using the asymptotic formulae for the Bessel functions for large arguments when ν is fixed (see, e.g., [65]), we can see that for the function $h_\nu(u)$ from Eq. (20.14) the condition (4.5) is satisfied if $z + z' + |\Delta t| < 2z_b$. As for

$|u| < k$ one has $h_\nu(ue^{\pi i}) = -h_\nu(u)$, the condition (4.12) is also satisfied for the function $h_\nu(u)$. Note that $h_\nu(u) \sim u^{1-\delta_{k0}}$ for $u \to 0$ and the residue term on the right of formula (4.14) vanishes. Applying to the sum over n in Eq. (20.13) formula (4.14), one obtains

$$\begin{aligned} W(x,x') &= \frac{k_D^{D-1}(zz')^{D/2}}{2^D \pi^{D-1}} \int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}} \\ & \left\{ \frac{1}{z_a^2} \int_0^\infty \frac{h_\nu(u)du}{\bar{J}_\nu^{(a)2}(u) + \bar{Y}_\nu^{(a)2}(u)} - \frac{2}{\pi} \int_k^\infty du\, u \frac{\Omega_{a\nu}(uz_a, uz_b)}{\sqrt{u^2-k^2}} \right. \\ & \left. \times G_\nu^{(a)}(uz_a, uz) G_\nu^{(a)}(uz_a, uz') \cosh(\Delta t \sqrt{u^2-k^2}) \right\}, \end{aligned} \tag{20.15}$$

where we use notations (4.15),(12.30). Note that we have assumed values of the coefficients A_j and B_j for which all zeros for Eq. (20.9) are real and have omitted the residue terms. In the following we will consider this case only.

In the way similar to that used before, it can be seen that the first integral in the figure braces in Eq. (20.15) is presented in the form

$$\begin{aligned} \int_0^\infty \frac{h_\nu(u)du}{\bar{J}_\nu^{(a)2}(u) + \bar{Y}_\nu^{(a)2}(u)} &= z_a^2 \int_0^\infty duu \frac{e^{-i\Delta t\sqrt{u^2+k^2}}}{\sqrt{x^2+k^2}} J_\nu(uz) J_\nu(uz') \\ & - \frac{2z_a^2}{\pi} \int_k^\infty du\ u \frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)} \frac{K_\nu(uz)K_\nu(uz')}{\sqrt{u^2-k^2}} \\ & \times \cosh(\Delta t\sqrt{u^2-k^2}). \end{aligned} \tag{20.16}$$

Substituting this into formula (20.15), the Wightman function can be written in the form [56]

$$\begin{aligned} W(x,x') &= W^{(a)}(x,x') - \frac{k_D^{D-1}(zz')^{D/2}}{2^{D-1}\pi^D} \int d\mathbf{k}\, e^{\mathbf{k}\Delta\mathbf{x}} \int_k^\infty duu \frac{\Omega_{a\nu}(uz_a, uz_b)}{\sqrt{u^2-k^2}} \\ & \times G_\nu^{(a)}(uz_a, uz) G_\nu^{(a)}(uz_a, uz') \cosh(\Delta t\sqrt{u^2-k^2}). \end{aligned} \tag{20.17}$$

Here the term

$$W^{(a)}(x,x') = W_{\mathrm{S}}(x,x') + \langle \varphi(x)\varphi(x') \rangle_a \tag{20.18}$$

does not depend on the parameters of the boundary at $y = b$ and is the Wightman function for a single brane located at $y = a$. In formula (20.18), the term

$$W_{\mathrm{S}}(x,x') = \frac{k_D^{D-1}(zz')^{D/2}}{2^D\pi^{D-1}} \int d\mathbf{k}\, e^{\mathbf{k}\Delta\mathbf{x}} \int_0^\infty du\, u \frac{e^{-i\Delta t\sqrt{u^2+k^2}}}{\sqrt{u^2+k^2}} J_\nu(uz) J_\nu(uz'), \tag{20.19}$$

does not depend on the boundary conditions and is the Wightman function for the AdS space without boundaries, and the second term on the right,

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle_a &= -\frac{k_D^{D-1}(zz')^{D/2}}{2^{D-1}\pi^D} \int d\mathbf{k}\, e^{\mathbf{k}\Delta\mathbf{x}} \int_k^\infty du\, u \frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)} \\ & \times \frac{K_\nu(uz)K_\nu(uz')}{\sqrt{u^2-k^2}} \cosh(\Delta t\sqrt{u^2-k^2}), \end{aligned} \tag{20.20}$$

is induced in the region $z > z_a$ by a single brane at $z = z_a$ when the brane $z = z_b$ is absent.

The Wightman function in the region $z_a \leqslant z \leqslant z_b$ can also be presented in the equivalent form

$$\begin{aligned} W(x,x') &= W^{(b)}(x,x') - \frac{k_D^{D-1}(zz')^{D/2}}{2^{D-1}\pi^D} \int d\mathbf{k}\, e^{\mathbf{k}\Delta\mathbf{x}} \int_k^\infty du\, u \frac{\Omega_{b\nu}(uz_a, uz_b)}{\sqrt{u^2-k^2}} \\ &\quad \times G_\nu^{(b)}(uz_b, uz) G_\nu^{(b)}(uz_b, uz') \cosh(\Delta t\sqrt{u^2-k^2}). \end{aligned} \tag{20.21}$$

In this formula

$$W^{(b)}(x,x') = W_{\mathrm{S}}(x,x') + \langle\varphi(x)\varphi(x')\rangle_b, \tag{20.22}$$

where

$$\begin{aligned} \langle\varphi(x)\varphi(x')\rangle_b &= -\frac{k_D^{D-1}(zz')^{D/2}}{2^{D-1}\pi^D} \int d\mathbf{k}\, e^{\mathbf{k}\Delta\mathbf{x}} \int_k^\infty du\, u \frac{\bar{K}_\nu^{(b)}(uz_b)}{\bar{I}_\nu^{(b)}(uz_b)} \\ &\quad \times \frac{I_\nu(uz)I_\nu(uz')}{\sqrt{u^2-k^2}} \cosh(\Delta t\sqrt{u^2-k^2}), \end{aligned} \tag{20.23}$$

is the Wightman function for a single brane at $y = b$. The term (20.23) is the boundary part induced in the region $z < z_b$ by the brane at $y = b$. Combining two forms, formulae (20.17) and (20.21), we see that the expressions for the Wightman function in the region $z_a \leqslant z \leqslant z_b$ is symmetric under the interchange $a \rightleftarrows b$ and $I_\nu \rightleftarrows K_\nu$. Note that the expression for the Wightman function is not symmetric with respect to the interchange of the brane indices. The reason for this is that, though the background AdS spacetime is homogeneous, the boundaries have nonzero extrinsic curvature tensors and two sides of the boundaries are not equivalent. In particular, for the geometry of a single brane the VEVs differ for the regions on the left and on the right of the brane. Here the situation is similar to that for the case of a spherical shell on background of the Minkowski spacetime.

20.2 Casimir densities for a single brane

In this subsection we will consider the VEV of the EMT for a scalar field in the case of a single brane located at $z = z_a$. As it has been shown in the previous subsection the Wightman function for this geometry is presented in the form (20.18). The brane induced part $\langle\varphi(x)\varphi(x')\rangle_a$ is given by formula (20.20) in the region $z > z_a$ and by formula (20.23) with replacement $z_b \to z_a$ in the region $z < z_a$. For points away from the brane this part is finite in the coincidence limit and in the corresponding formulae for the Wightman function we can directly put $x = x'$. Introducing a new integration variable $v = \sqrt{u^2-k^2}$, transforming to polar coordinates in the plane (v,k) and integrating over angular part, the following formula can be derived

$$\int_0^\infty dk k^{D-2} \int_k^\infty du \frac{uf(u)}{\sqrt{u^2-k^2}} = \frac{\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)}{2\Gamma(D/2)} \int_0^\infty du u^{D-1} f(u). \tag{20.24}$$

By using this formula and Eq. (20.20), the boundary induced VEV for the field square in the region $z > z_a$ is presented in the form [56]

$$\langle\varphi^2\rangle_{\mathrm{b}}^{(a)} = -\frac{k_D^{D-1}z^D}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty du\, u^{D-1} \frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)} K_\nu^2(uz), \quad z > z_a. \tag{20.25}$$

The corresponding formula in the region $z < z_a$ is obtained from Eq. (20.23) by a similar way and differs from Eq. (20.25) by replacements $I_\nu \rightleftarrows K_\nu$.

The VEV of the EMT can be evaluated by substituting expressions for the positive frequency Wightman function and VEV of the field square into Eq. (9.10). First of all we will consider the region $z > z_a$. The vacuum EMT is diagonal and can be presented in the form

$$\langle 0|T_i^k|0\rangle = \langle 0|T_i^k|0\rangle_{\rm S} + \langle T_i^k\rangle_a, \tag{20.26}$$

where

$$\langle 0|T_i^k|0\rangle_{\rm S} = \frac{k_D^{D+1}\delta_i^k}{2^D\pi^{D/2}}\Gamma\left(1-\frac{D}{2}\right)\int_0^\infty du u^{D-1} f^{(i)}[J_\nu(u)], \tag{20.27}$$

is the VEV for the EMT in the AdS background without boundaries, and the term [56]

$$\langle T_i^k\rangle_a = -\frac{k_D^{D+1}z^D\delta_i^k}{2^{D-1}\pi^{D/2}\Gamma(D/2)}\int_0^\infty du u^{D-1}\frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)}F^{(i)}[K_\nu(uz)], \tag{20.28}$$

is induced in the region $z > z_a$ by a single boundary at $z = z_a$. For a given function $g(v)$ the functions $F^{(i)}[g(v)]$ in formula (20.28) are defined as

$$\begin{aligned} F^{(i)}[g(v)] &= \left(\frac{1}{2}-2\zeta\right)\left[v^2 g'^2(v) + \left(D + \frac{4\zeta}{4\zeta-1}\right) v g(v) g'(v) + \right. \\ &\quad \left. + \left(\nu^2 + v^2 + \frac{2v^2}{D(4\zeta-1)}\right) g^2(v)\right], \quad i = 0, 1, \ldots, D-1, \end{aligned} \tag{20.29}$$

$$\begin{aligned} F^{(D)}[g(v)] &= -\frac{v^2}{2} g'^2(v) + \frac{D}{2}\left(4\zeta - 1\right) v g(v) g'(v) + \\ &\quad + \frac{1}{2}\left[v^2 + \nu^2 + 2\zeta D(D+1) - D^2/2\right] g^2(v), \end{aligned} \tag{20.30}$$

and the expressions for the functions $f^{(i)}[g(v)]$ are obtained from those for $F^{(i)}[g(v)]$ by the replacement $v \to iv$. Note that the boundary-induced part (20.28) is finite for points away the brane and, hence, the renormalization procedure is needed for the boundary-free part only. The latter is well investigated in literature.

In a similar way, for the VEVs induced by a single brane in the region $z < z_a$, by making use of expression (20.23) (with replacement $z_b \to z_a$), one obtains

$$\langle T_i^k\rangle_a = -\frac{k_D^{D+1}z^D\delta_i^k}{2^{D-1}\pi^{D/2}\Gamma(D/2)}\int_0^\infty du u^{D-1}\frac{\bar{K}_\nu^{(a)}(uz_a)}{\bar{I}_\nu^{(a)}(uz_a)}F^{(i)}[I_\nu(uz)]. \tag{20.31}$$

Note that VEVs (20.28), (20.31) depend only on the ratio z/z_a which is related to the proper distance from the brane by the equation

$$z/z_a = e^{k_D(y-a)}. \tag{20.32}$$

As we see, for the part of the EMT corresponding to the coordinates in the hyperplane parallel to the branes one has $\langle T_{\mu\nu}\rangle^{(a)} \sim \eta_{\mu\nu}$. Of course, we could expect this result from the problem symmetry. It can be seen that the VEVs obtained above obey the continuity equation $\nabla_k T_i^k = 0$, which for the AdS metric takes the form

$$z^{D+1}\partial_z\left(z^{-D}T_D^D\right) + DT_0^0 = 0. \tag{20.33}$$

For a conformally coupled massless scalar $\nu = 1/2$, and by making use of the expressions for the modified Bessel functions, it can be seen that $\langle T_i^k\rangle^{(a)} = 0$ in the region $z > z_a$ and

$$\langle T_D^D\rangle^{(a)} = -D\langle T_0^0\rangle^{(a)} = -\frac{(k_D z/z_a)^{D+1}}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty \frac{t^D\, dt}{\frac{B_a(t-1)+2A_a}{B_a(t+1)-2A_a}e^t + 1} \tag{20.34}$$

in the region $z < z_a$. Note that the corresponding energy-momentum tensor for a single Robin plate in the Minkowski bulk vanishes [32] and the result for the region $z > z_a$ is obtained by a simple conformal transformation from that for the Minkowski case. In the region $z < z_a$ this procedure does not work as in the AdS problem one has $0 < z < z_a$ instead of $-\infty < z < z_a$ in the Minkowski problem and, hence, the part of AdS under consideration is not a conformal image of the corresponding manifold in the Minkowski spacetime.

The brane-induced VEVs given by equations (20.28) and (20.31), in general, can not be further simplified and need numerical calculations. Relatively simple analytic formulae can be obtained in limiting cases. First of all, as a partial check, in the limit $k_D \to 0$ the corresponding formulae for a single plate on the Minkowski bulk are obtained (see Ref. [32]). This can be seen noting that in this limit $\nu \sim m/k_D$ is large and by introducing the new integration variable in accordance with $u = \nu y$, we can replace the modified Bessel functions by their uniform asymptotic expansions for large values of the order. The Minkowski result is obtained in the leading order.

In the limit $z \to z_a$ for a fixed k_D expressions (20.28) and (20.31) diverge. In accordance with (20.32), this corresponds to small proper distances from the brane. Near the brane the main contribution into the integral over u in Eqs. (20.28), (20.31) comes from large values of u and we can replace the modified Bessel functions by their asymptotic expressions for large values of the argument when the order is fixed (see, for instance, [65]). To the leading order this yields

$$\langle T_0^0 \rangle^{(a)} \approx \frac{\langle T_D^D \rangle^{(a)}}{1 - z_a/z} \approx -\Gamma\left(\frac{D+1}{2}\right) \frac{D k_D^{D+1} (\zeta - \zeta_D) k_a}{2^D \pi^{(D+1)/2} |1 - z_a/z|^{D+1}}, \tag{20.35}$$

where k_a is defined after formula (19.22). Note that the leading terms for the components with $i = 0, 1, \ldots, D-1$ are symmetric with respect to the brane, and the ${}_D^D$ – component has opposite signs for different sides of the brane. Near the brane the vacuum energy densities have opposite signs for the cases of Dirichlet ($B_a = 0$) and non-Dirichlet ($B_a \neq 0$) boundary conditions. Recall that for a conformally coupled massless scalar the vacuum EMT vanishes in the region $z > z_a$ and is given by expression (20.34) in the region $z < z_a$. The latter is finite everywhere including the points on the brane. For large proper distances from the brane compared with the AdS curvature radius, $k_D|y - a| \gg 1$, the boundary induced EMT vanishes as $\exp[2\nu k_D(a - y)]$ in the region $y > a$ and as $\exp[k_D(2\nu + D)(y - a)]$ in the region $y < a$. The same behavior takes place for a fixed $y - a$ and large values of the parameter k_D. In the large mass limit, $m \gg k_D$, the boundary parts are exponentially suppressed.

20.3 Two-brane geometry

In this subsection we will investigate the VEVs for the field square and the EMT in the region between two branes. Taking the coincidence limit in the formulae for the Wightman function and using formula (20.24), for the VEV of the field square one finds [56]

$$\begin{aligned} \langle 0|\varphi^2|0\rangle &= \langle 0|\varphi^2|0\rangle_{\mathrm{S}} + \langle \varphi^2 \rangle_j - \frac{k_D^{D-1} z^D}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \\ &\quad \times \int_0^\infty du\, u^{D-1} \Omega_{j\nu}(uz_a, uz_b) G_\nu^{(j)2}(uz_j, uz), \end{aligned} \tag{20.36}$$

where $j = a, b$ provide two equivalent representations. The last term on the right of this formula is finite for points on the brane at $y = j$ and diverges on the other brane. In the similar way, substituting the corresponding Wightman function from Eq. (20.17) into the mode-sum formula

(9.10), we obtain

$$\begin{aligned}\langle 0|T_i^k|0\rangle &= \langle 0|T_i^k|0\rangle_{\rm S}+\langle T_i^k\rangle_j-\frac{k_D^{D+1}z^D\delta_i^k}{2^{D-1}\pi^{D/2}\Gamma(D/2)}\\ &\times\int_0^\infty du u^{D-1}\Omega_{j\nu}(uz_a,uz_b)F^{(i)}[G_\nu^{(j)}(uz_j,uz)],\end{aligned}\tag{20.37}$$

with the functions $F^{(i)}[g(v)]$ from Eqs. (20.29), (20.30). The formulae for the cases $j=a,b$ are obtained from each other by the replacements $a\rightleftarrows b$, $I_\nu\rightleftarrows K_\nu$. The VEVs in the region between the branes are not symmetric under the interchange of indices of the branes. As it has been mentioned above, the reason for this is that, though the background spacetime is homogeneous, due to the non-zero extrinsic curvature tensors for the branes, the regions on the left and on the right of the brane are not equivalent. By the same way as for the case of a single brane, it can be seen that in the limit $k_D\to 0$ from formulae (20.36) and (20.37) the corresponding results for two plates geometry in the Minkowski bulk. For a conformally coupled massless scalar field $\nu=1/2$ and the formulae for the VEVs in the region between the branes can also be obtained from the corresponding formulae for parallel Robin plates in the Minkowski bulk by conformal transformation [140].

Now we turn to the interaction forces between the branes. The vacuum force acting per unit surface of the brane at $z=z_j$ is determined by the $_D^D$ – component of the vacuum EMT at this point. The corresponding effective pressures can be presented as the sum $p^{(j)}=p_1^{(j)}+p_{(\rm int)}^{(j)}$, $j=a,b$, where the first term on the right is the pressure for a single brane at $z=z_j$ when the second brane is absent. This term is divergent due to the surface divergences in the VEVs. The second term is determined by the last term on the rhs in (20.37) evaluated at $z=z_j$. This term is the pressure induced by the presence of the second brane. Using the relations $G_\nu^{(j)}(u,u)=-B_j$, $G_\nu^{(j)\prime}(u,u)=A_j/u$, one finds

$$\begin{aligned}p_{(\rm int)}^{(j)} &= \frac{k_D^{D+1}}{2^D\pi^{D/2}\Gamma\left(\frac{D}{2}\right)}\int_0^\infty dx\, x^{D-1}\Omega_{j\nu}\left(xz_a/z_j,xz_b/z_j\right)\\ &\times\left[\left(x^2-\nu^2+2m^2/k_D^2\right)B_j^2-D(4\zeta-1)A_jB_j-A_j^2\right].\end{aligned}\tag{20.38}$$

In dependence of the values for the coefficients in the boundary conditions, these effective pressures can be either positive or negative, leading to repulsive or attractive forces. Note that due to the asymmetry in the VEV of the EMT, the interaction forces acting on the branes are not symmetric under the interchange of the brane indices. It can be seen that the vacuum effective pressures are negative for Dirichlet scalar and for a scalar with $A_a=A_b=0$ and, hence, the corresponding interaction forces are attractive for all values of the interbrane distance.

Let us consider the limiting cases for the interaction forces described by Eq. (20.38). For small distances compared with the AdS curvature radius, $k_D(b-a)\ll 1$, the leading terms are the same as for the plates in Minkowski bulk. In particular, in this limit the interaction forces are repulsive for Dirichlet boundary condition on one brane and non-Dirichlet boundary condition on the other, and are attractive for all other cases. For large distances between the branes, $k_D(b-a)\gg 1$ (this limit is realized in the Randall-Sundrum model), by using the expressions for the modified Bessel functions for small values of the argument, one can see that $p_{(\rm int)}^{(a)}\sim(z_a/2z_b)^{D+2\nu}$ and $p_{(\rm int)}^{(b)}\sim(z_a/2z_b)^{2\nu}$.

Now we consider the application of the results described in this section to the Randall-Sundrum braneworld models [138] based on the AdS geometry with one extra dimension. The fifth dimension y is compactified on an orbifold, S^1/Z_2 of length L, with $-L\leqslant y\leqslant L$. The orbifold fixed points at $y=a=0$ and $y=b$ are the locations of two 3-branes. We will allow

these submanifolds to have an arbitrary dimension D. The metric in the Randall-Sundrum model has the form (20.1) with $\sigma(y) = k_D|y|$. The corresponding boundary conditions for an untwisted scalar field are in form (20.3) with [56, 89]

$$\tilde{A}_j/\tilde{B}_j = -(n^{(j)}c_j + 4D\zeta k_D)/2, \tag{20.39}$$

and respectively

$$A_j/B_j = \left[D(1-4\zeta) - n^{(j)}c_j/k_D\right]/2, \tag{20.40}$$

where c_j are the brane mass terms. For a twisted scalar $\tilde{B}_j = 0$, which corresponds to Dirichlet boundary conditions on both branes. Recently the EMT in the Randall–Sundrum braneworld for a bulk scalar with zero brane mass terms c_1 and c_2 is considered in [141], where a general formula is given for the unrenormalized VEV in terms of the differential operator acting on the Green function. In our approach the application of the GAPF allowed to extract manifestly the part due to the AdS bulk without boundaries and for the points away from the boundaries the renormalization procedure is the same as for the boundary-free parts. In addition, the boundary-induced parts are presented in terms of exponentially convergent integrals convenient for numerical calculations.

From the point of view of embedding the Randall-Sundrum type braneworld models into a more fundamental theory, such as string/M theory, one may expect that a more complete version of this scenario must admit the presence of additional extra dimensions compactified on an internal manifold. The results discussed in this section can be generalized for the geometry of two parallel branes of codimension one on background of $(D+1)$ -dimensional spacetime with topology $AdS_{D_1+1} \times \Sigma$ and the line element

$$ds^2 = e^{-2k_D y}\eta_{\mu\sigma}dx^\mu dx^\sigma - e^{-2k_D y}\gamma_{ik}dX^i dX^k - dy^2, \tag{20.41}$$

where $\eta_{\mu\sigma} = \mathrm{diag}(1, -1, \ldots, -1)$ is the metric tensor for D_1-dimensional Minkowski spacetime $R^{(D_1-1,1)}$, and the coordinates X^i, $i = 1, \ldots, D_2$, cover the manifold Σ. The quantum effective potential and the problem of moduli stabilization in the orbifolded version of this model with zero mass parameters on the branes were discussed recently in [142]. In particular, it has been shown that one-loop effects induced by bulk scalar fields generate a suitable effective potential which can stabilize the hierarchy between the gravitational and electroweak scales. In [57] the Wightman function, the VEVs of the field square and the EMT for a scalar field brought about by the presence of two parallel infinite branes, located at $y = a$ and $y = b$ with boundary conditions (20.3), are investigated in a way similar to that used in this section.

21 Casimir densities for spherical branes in Rindler-like spacetimes

In previous section we have considered braneworld models on background of AdS spacetime. It seems interesting to generalize the study of quantum effects to other types of bulk spacetimes. In particular, bulk geometries generated by higher-dimensional black holes are of special interest. In these models, the tension and the position of the brane are tuned in terms of black hole mass and cosmological constant and brane gravity trapping occurs in just the same way as in the Randall-Sundrum model. Though, in the generic black hole background the investigation of brane-induced quantum effects is technically complicated, the exact analytical results can be obtained in the near horizon and large mass limit when the brane is close to the black hole horizon. In this limit the black hole geometry may be approximated by the Rindler-like manifold (for some investigations of quantum effects on background of Rindler-like spacetimes

see [143, 144] and references therein). In this section, by using the GAPF, we investigate the Wightman function, the VEVs of the field square and the EMT for a scalar field with an arbitrary curvature coupling parameter for two spherical branes in the bulk $Ri \times S^{D-1}$, where Ri is a two-dimensional Rindler spacetime [58].

21.1 Wightman function

Consider a scalar field $\varphi(x)$ propagating on background of $(D+1)$-dimensional Rindler-like spacetime $Ri \times S^{D-1}$. The corresponding metric is described by the line element

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - r_H^2 d\Sigma_{D-1}^2, \tag{21.1}$$

with the Rindler-like (τ, ξ) part and $d\Sigma_{D-1}^2$ is the line element for the space with positive constant curvature with the Ricci scalar $R = n(n+1)/r_H^2$, $n = D-2$. Line element (21.1) describes the near horizon geometry of $(D+1)$-dimensional topological black hole with the line element

$$ds^2 = A_H(r)dt^2 - \frac{dr^2}{A_H(r)} - r^2 d\Sigma_{D-1}^2, \tag{21.2}$$

where $A_H(r) = k + r^2/l^2 - r_0^D/l^2 r^n$ and the parameter k classifies the horizon topology, with $k = 0, -1, 1$ corresponding to flat, hyperbolic, and elliptic horizons, respectively. The parameter l is related to the bulk cosmological constant and the parameter r_0 depends on the mass of the black hole. In the non-extremal case the function $A_H(r)$ has a simple zero at $r = r_H$, and in the near horizon limit, introducing new coordinates τ and ρ in accordance with

$$\tau = A_H'(r_H)t/2, \quad r - r_H = A_H'(r_H)\xi^2/4, \tag{21.3}$$

the line element is written in the form (21.1). Note that for a $(D+1)$-dimensional Schwarzschild black hole one has $A_H(r) = 1 - (r_H/r)^n$ and, hence, $A_H'(r_H) = n/r_H$.

We will assume that the field satisfies Robin boundary conditions (19.1) on the hypersurfaces $\xi = a$ and $\xi = b$, $a < b$. In accordance with (21.3), the hypersurface $\xi = j$ corresponds to the spherical shell near the black hole horizon with the radius $r_j = r_H + A_H'(r_H)j^2/4$. In the corresponding braneworld scenario based on the orbifolded version of the model the region between the branes is employed only and the ratio $\tilde{A}_j/\tilde{B}_j$ for untwisted bulk scalars is related to the brane mass parameters c_j of the field by the formula [144]

$$\tilde{A}_j/\tilde{B}_j = \frac{1}{2}\left(c_j - \zeta/j\right), \; j = a, b. \tag{21.4}$$

For a twisted scalar Dirichlet boundary conditions are obtained on both branes.

In hyperspherical coordinates (see Section 12 for the notations) the eigenfunctions in the region between the branes can be written in the form

$$\varphi_\sigma(x) = \beta_\sigma G_{i\omega}^{(b)}(\lambda_l b, \lambda_l \xi) Y(m_k; \vartheta, \phi) e^{-i\omega\tau}, \tag{21.5}$$

where the function $G_\nu^{(j)}(x, y)$ is defined by formula (12.30) with the barred notations (4.2), where $A_j = \tilde{A}_j$, $B_j = \tilde{B}_j/j$, and

$$\lambda_l = \frac{1}{r_H}\sqrt{l(l+n) + \zeta n(n+1) + m^2 r_H^2}\,. \tag{21.6}$$

From the boundary conditions we find that the eigenvalues for ω are roots to the equation $Z_{i\omega}(\lambda_l a, \lambda_l b) = 0$, where the function $Z_{i\omega}(u, v)$ is defined by formula (5.14), and, hence, $\omega = \Omega_s = \Omega_s(\lambda_l a, \lambda_l b)$, $s = 1, 2, \ldots$. For the normalization coefficient one has

$$\beta_\sigma^2 = \left. \frac{r_H^{1-D} \bar{I}_{i\omega}^{(a)}(\lambda_l a)}{N(m_k)\bar{I}_{i\omega}^{(b)}(\lambda_l b)\partial_\omega Z_{i\omega}(\lambda_l a, \lambda_l b)} \right|_{\omega=\Omega_s} . \tag{21.7}$$

Substituting eigenfunctions (21.5) into the mode-sum formula, for the Wightman function in the region between the branes one finds

$$\begin{aligned} W(x, x') &= \frac{r_H^{1-D}}{nS_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \sum_{s=1}^{\infty} \frac{\bar{I}_{i\omega}^{(a)}(\lambda_l a) e^{-i\omega\Delta\tau}}{\bar{I}_{i\omega}^{(b)}(\lambda_l b)\partial_\omega Z_{i\omega}(\lambda_l a, \lambda_l b)} \\ &\quad \times \left. G_{i\omega}^{(b)}(\lambda_l b, \lambda_l \xi) G_{i\omega}^{(b)}(\lambda_l b, \lambda_l \xi') \right|_{\omega=\Omega_s}, \end{aligned} \tag{21.8}$$

with $\Delta\tau = \tau - \tau'$. The application of summation formula (5.18) to the sum over s leads to the result

$$\begin{aligned} W(x, x') &= W_0(x, x') + \langle \varphi(x)\varphi(x') \rangle^{(j)} - \frac{r_H^{1-D}}{\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \\ &\quad \times \int_0^\infty d\omega\, \Omega_{j\omega}(\lambda_l a, \lambda_l b) G_\omega^{(j)}(\lambda_l j, \lambda_l \xi) G_\omega^{(j)}(\lambda_l j, \lambda_l \xi') \cosh(\omega\Delta\tau), \end{aligned} \tag{21.9}$$

with $j = b$, where $W_0(x, x')$ is the Wightman function for the geometry without boundaries and the part

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle^{(b)} &= -\frac{r_H^{1-D}}{\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \int_0^\infty d\omega \frac{\bar{K}_\omega^{(b)}(\lambda_l b)}{\bar{I}_\omega^{(b)}(\lambda_l b)} \\ &\quad \times I_\omega(\lambda_l \xi) I_\omega(\lambda_l \xi') \cosh(\omega\Delta\tau) \end{aligned} \tag{21.10}$$

is induced in the region $\xi < b$ by the presence of the brane at $\xi = b$. Note that the representation (21.9) is valid under the condition $a^2 e^{|\Delta\tau|} < \xi\xi' < b^2 e^{|\Delta\tau|}$. As it has been shown in [144], the Wightman function for the boundary-free geometry may be written in the form

$$\begin{aligned} W_0(x, x') &= \tilde{W}_0(x, x') - \frac{r_H^{1-D}}{\pi^2 n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \\ &\quad \times \int_0^\infty d\omega e^{-\omega\pi} \cos(\omega\Delta\tau) K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi'), \end{aligned} \tag{21.11}$$

where $\tilde{W}_0(x, x')$ is the Wightman function for the bulk geometry $R^2 \times S^{D-1}$. Outside the horizon the divergences in the coincidence limit of the expression on the right of (21.11) are contained in the first term. It can be seen that the Wightman function in the region between the branes can also be presented in the form (21.9) with $j = a$. In this representation $\langle \varphi(x)\varphi(x') \rangle^{(a)}$ is induced in the region $\xi > a$ by the presence of the brane at $\xi = a$ and the corresponding expression is obtained from (21.10) by the replacements $b \to a$, $I_\omega \rightleftarrows K_\omega$.

21.2 Vacuum densities

In the coincidence limit, from the formulae for the Wightman function one obtains two equivalent forms for the VEV of the field square:

$$\begin{aligned}\langle 0|\varphi^2|0\rangle &= \langle 0_0|\varphi^2|0_0\rangle + \langle\varphi^2\rangle^{(j)} \\ &\quad -\frac{r_H^{1-D}}{\pi S_D}\sum_{l=0}^{\infty} D_l \int_0^{\infty} d\omega\, \Omega_{j\omega}(\lambda_l a, \lambda_l b) G_\omega^{(j)2}(\lambda_l j, \lambda_l \xi),\end{aligned} \tag{21.12}$$

corresponding to $j = a$ and $j = b$, and $|0_0\rangle$ is the amplitude for the vacuum without boundaries. The coefficient D_l in this formula is defined by relation (12.14). The VEV $\langle 0_0|\varphi^2|0_0\rangle$ is obtained from the corresponding Wightman function given by (21.11). For the points outside the horizon, the renormalization procedure is needed for the first term on the right only, which corresponds to the VEV in the geometry $R^2 \times S^{D-1}$. This procedure is realized in [144] on the base of the zeta function technique.

In (21.12), the part $\langle\varphi^2\rangle^{(j)}$ is induced by a single brane at $\xi = j$ when the second brane is absent. For the geometry of a single brane at $\xi = b$, from (21.10) one has

$$\langle\varphi^2\rangle^{(b)} = -\frac{r_H^{1-D}}{\pi S_D}\sum_{l=0}^{\infty} D_l \int_0^{\infty} d\omega \frac{\bar{K}_\omega^{(b)}(\lambda_l b)}{\bar{I}_\omega^{(b)}(\lambda_l b)} I_\omega^2(\lambda_l \xi). \tag{21.13}$$

The expression for $\langle\varphi^2\rangle^{(a)}$ is obtained from (21.13) by the replacements $b \to a$, $I_\omega \rightleftarrows K_\omega$. The last term on the right of formula (19.13) is induced by the presence of the second brane. It is finite on the brane at $\xi = j$ and diverges for points on the other brane. By taking into account the relation $G_\omega^{(j)}(u,u) = B_j/j$, we see that for the Dirichlet boundary condition this term vanishes on the brane $\xi = j$.

By using the formulae for the Wightman function and the VEV of the field square, one obtains two equivalent forms for the VEV of the EMT, corresponding to $j = a$ and $j = b$ (no summation over i):

$$\begin{aligned}\langle 0|T_i^k|0\rangle &= \langle 0_0|T_i^k|0_0\rangle + \langle T_i^k\rangle^{(j)} - \delta_i^k \frac{r_H^{1-D}}{\pi S_D}\sum_{l=0}^{\infty} D_l \lambda_l^2 \\ &\quad \times \int_0^{\infty} d\omega\, \Omega_{j\omega}(\lambda_l a, \lambda_l b) F^{(i)}[G_\omega^{(j)}(\lambda_l j, \lambda_l \xi)].\end{aligned} \tag{21.14}$$

The functions $F^{(i)}[g(z)]$ in this formula are defined by relations (18.15) and (18.16) with $\lambda \to \lambda_l$ and $g(z) = G_\omega^{(j)}(\lambda_l j, z)$. In formula (21.14),

$$\langle 0_0|T_i^k|0_0\rangle = \delta_i^k \frac{r_H^{1-D}}{\pi^2 S_D}\sum_{l=0}^{\infty} D_l \lambda_l^2 \int_0^{\infty} d\omega \sinh \pi\omega\, f^{(i)}[K_{i\omega}(\lambda_l \xi)], \tag{21.15}$$

with $f^{(i)}[g(z)]$ defined by Eq. (18.15), is the corresponding VEV for the vacuum without boundaries. The term $\langle T_i^k\rangle^{(j)}$ is induced by the presence of a single spherical brane located at $\xi = j$. For the brane at $\xi = b$ and in the region $\xi < b$ one has (no summation over i)

$$\langle T_i^k\rangle^{(b)} = -\delta_i^k \frac{r_H^{1-D}}{\pi S_D}\sum_{l=0}^{\infty} D_l \lambda_l^2 \int_0^{\infty} d\omega \frac{\bar{K}_\omega^{(b)}(\lambda_l b)}{\bar{I}_\omega^{(b)}(\lambda_l b)} F^{(i)}[I_\omega(\lambda_l \xi)]. \tag{21.16}$$

For the geometry of a single brane at $\xi = a$, the corresponding expression in the region $\xi > a$ is obtained from (21.16) by the replacements $b \to a$, $I_\omega \rightleftarrows K_\omega$. It can be easily seen that for a conformally coupled massless scalar the boundary induced part in the EMT is traceless.

Vacuum forces acting on the branes are presented in the form of the sum of self-action and interaction terms. The vacuum effective pressures corresponding to the interaction forces are obtained from the third summand on the right of Eq. (21.14), taking $i = k = 1$, $\xi = j$ [58]:

$$p^{(j)}_{(\mathrm{int})} = \frac{\tilde{A}_j^2}{2j^2}\frac{r_H^{1-D}}{\pi S_D}\sum_{l=0}^{\infty} D_l \int_0^{\infty} d\omega \left[\left(\lambda_l^2 j^2 + \omega^2\right)\beta_j^2 + 4\zeta\beta_j - 1\right] \Omega_{j\omega}(\lambda_l a, \lambda_l b), \tag{21.17}$$

with $\beta_j = \tilde{B}_j / j\tilde{A}_j$. The interaction force acts on the surface $\xi = a+0$ for the brane at $\xi = a$ and on the surface $\xi = b-0$ for the brane at $\xi = b$. In dependence of the values for the coefficients in the boundary conditions, effective pressures (19.23) can be either positive or negative, leading to repulsive or attractive forces, respectively. For Dirichlet or Neumann boundary conditions on both branes the interaction forces are always attractive. For Dirichlet boundary condition on one brane and Neumann boundary condition on the other one has $p^{(j)}_{(\mathrm{int})} > 0$ and the interaction forces are repulsive for all distances between the branes. The investigation of the vacuum densities in various asymptotic regions of the parameters can be forund in [58, 144].

22 Radiation from a charge moving along a helical orbit inside a dielectric cylinder

The radiation from a charged particle moving along a helical orbit in vacuum has been widely discussed in literature. This type of electron motion is used in helical undulators for generating electromagnetic radiation in a narrow spectral interval at frequencies ranging from radio or millimeter waves to X-rays (see, for instance, [145]). The unique characteristics, such as high intensity and high collimation, have resulted in extensive applications of this radiation in a wide variety of experiments and in many disciplines. In this section we apply the GAPF for the investigation of the radiation on the lowest azimuthal mode by a charged particle moving along a helical orbit inside a dielectric cylinder [59].

Consider a dielectric cylinder of radius ρ_1 and dielectric permittivity ε_0 and a point charge q moving along the helical trajectory of radius $\rho_0 < \rho_1$. We assume that the system is immersed in a homogeneous medium with permittivity ε_1. The velocities of the charge along the axis of the cylinder and in the perpendicular plane we will denote by $v_\parallel$ and $v_\perp$, respectively. In a properly chosen cylindrical coordinate system (ρ, ϕ, z) with the z-axis along the cylinder axis, the vector potential of the electromagnetic field is presented in the form of the Fourier expansion

$$A_l(\mathbf{r}, t) = \sum_{m=-\infty}^{\infty} e^{im(\phi-\omega_0 t)} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-v_\parallel t)} A_{ml}(k_z, \rho), \tag{22.1}$$

where $\omega_0 = v_\perp/\rho_0$ is the angular velocity of the charge. The radiation field inside the dielectric cylinder is presented as a sum over the eigenmodes of the cylinder. Unlike to the case of the waveguide with perfectly conducting walls, here the eigenmodes with $m \neq 0$ are not decomposed into independent transverse electric (TE or M-type) and transverse magnetic (TM or E-type) parts. This decomposition takes place only for the $m = 0$ mode. The radiation on this mode is present under the condition

$$\beta_{1\parallel} < 1 < \beta_{0\parallel},\ \beta_{i\parallel} = v_\parallel\sqrt{\varepsilon_i}/c,\ i = 1, 2. \tag{22.2}$$

With these conditions the equation for the corresponding TM eigenmodes for k_z has the form

$$\varepsilon_0|\lambda_1|\frac{J_0'(\lambda_0\rho_1)}{J_0(\lambda_0\rho_1)} + \varepsilon_1\lambda_0\frac{K_0'(|\lambda_1|\rho_1)}{K_0(|\lambda_1|\rho_1)} = 0,\ \lambda_i = k_z\sqrt{\beta_{i\parallel}^2 - 1}. \tag{22.3}$$

The corresponding eigenvalues for k_z we will denote by $\pm k_s^{(\mathrm{E})}$, $s = 1, 2, \ldots$. The eigenmodes of the M-type are solutions of the equation

$$|\lambda_1| \frac{J_0'(\lambda_0 \rho_1)}{J_0(\lambda_0 \rho_1)} + \lambda_0 \frac{K_0'(|\lambda_1|\rho_1)}{K_0(|\lambda_1|\rho_1)} = 0, \tag{22.4}$$

and we will denote them by $k_z = \pm k_s^{(\mathrm{M})}$, $s = 1, 2, \ldots$.

For the radiation intensities on the mode $m = 0$ inside the dielectric waveguide we have

$$I_{m=0} = I_{m=0}^{(\mathrm{E})} + I_{m=0}^{(\mathrm{M})}, \tag{22.5}$$

where the contributions of the TM and TE modes are given by the formulae [59]

$$I_{m=0}^{(\mathrm{E})} = \frac{2q^2 v_\parallel}{\rho_1^2} \sum_s \frac{\beta_{0\parallel}^2 - 1}{\varepsilon_0 - \varepsilon_1} \frac{J_0^2(\rho_0 \lambda_s^{(\mathrm{E})})}{(\varepsilon_0/\varepsilon_1) J_0'^2(\rho_1 \lambda_s^{(\mathrm{E})}) + (\beta_{0\parallel}^2 - 1) J_0^2(\rho_1 \lambda_s^{(\mathrm{E})})}, \tag{22.6}$$

$$I_{m=0}^{(\mathrm{M})} = \frac{2q^2 v_\perp^2}{\rho_1^2 v_\parallel} \sum_s \frac{J_1^2(\rho_0 \lambda_s^{(\mathrm{M})})}{(\varepsilon_0 - \varepsilon_1) J_0'^2(\rho_1 \lambda_s^{(\mathrm{M})})}. \tag{22.7}$$

Here we have introduced the notation

$$\lambda_s^{(\mathrm{F})} = k_s^{(\mathrm{F})} \sqrt{\beta_{0\parallel}^2 - 1}, \ \mathrm{F} = \mathrm{E}, \mathrm{M}. \tag{22.8}$$

In formulae (22.6) and (22.7) the upper limit for the summation over s, which we will denote by s_m, is determined by the dispersion law of the dielectric permittivity $\varepsilon_0 = \varepsilon_0(\omega)$ through the condition $\varepsilon_0(\omega_s^{(\mathrm{F})}) > c^2/v_\parallel^2$. The summands with a given s describe the radiation field with the frequency $\omega_s^{(\mathrm{F})} = v_\parallel k_s^{(\mathrm{F})}$. As the modes $k_s^{(\mathrm{F})}$ are not explicitly known as functions on s, formulae (22.6), (22.7) are not convenient for the evaluation of the corresponding radiation intensities. More convenient form may be obtained by making use of the GAPF.

Let us derive a summation formula for the series over zeros of the function

$$C_\alpha(\eta, z) = V_\alpha \{J_0(z), K_0(\eta z)\}, \tag{22.9}$$

where and in what follows for given functions $F(z)$ and $G(z)$ we use the notation

$$V\{F(z), G(\eta z)\} = F(z) G'(\eta z) + \alpha \eta F'(z) G(\eta z), \tag{22.10}$$

with $\alpha \geqslant 1$ and η being real constants. In the GAPF we substitute

$$g(z) = i f(z) \frac{V_\alpha \{Y_0(z), K_0(\eta z)\}}{C_\alpha(\eta, z)}. \tag{22.11}$$

For the combinations of the functions entering in the GAPF one has

$$f(z) - (-1)^k g(z) = f(z) \frac{V_\alpha \{H_0^{(k)}(z), K_0(\eta z)\}}{C_\alpha(\eta, z)}. \tag{22.12}$$

Let us denote positive zeros of the function $C_\alpha(\eta, z)$ by k_s, $s = 1, 2, \ldots$, assuming that these zeros are arranged in the ascending order. Note that, for $z \gg 1$ we have $C_\alpha(\eta, z) \approx K_0'(\eta z) - \alpha \eta z K_0(\eta z)/2 < 0$ and, hence, $k_s \gtrsim 1$. By using the asymptotic formulae for the cylinder

functions for large values of the argument, it can be seen that for large values s one has $k_s \approx -\arctan(1/\alpha\eta)+\pi/4+\pi s$. For the derivative of the function $C_\alpha(\eta,z)$ at the zeros k_s one obtains

$$C'_\alpha(\eta,z) = -\eta\frac{K_0(\eta z)}{J_0(z)}\left[\alpha(1+\alpha\eta^2)J_0'^2(z)+(\alpha-1)J_0^2(z)\right],\ z=k_s. \tag{22.13}$$

In particular, it follows from here that the zeros are simple. Assuming that the function $f(z)$ is analytic in the right half-plane, for the residue term in the GAPF one finds

$$\operatorname*{Res}_{z=k_s} g(z) = -\frac{i}{\pi}P_\alpha(k_s)f(k_s), \tag{22.14}$$

where we have introduced the notation

$$P_\alpha(z) = \frac{2\alpha/z}{\alpha(1+\alpha\eta^2)J_0'^2(z)+(\alpha-1)J_0^2(z)}. \tag{22.15}$$

Substituting the expressions for the separate terms into the GAPF, we obtain the following result [59]

$$\begin{aligned}\lim_{x_0\to\infty}\left[\sum_{s=1}^{s_0}P_\alpha(k_s)f(k_s)-\int_0^{x_0}dx\,f(x)\right] &= -\frac{1}{\pi}\int_0^\infty dx\left[f(ix)\frac{V_\alpha\{K_0(x),H_0^{(2)}(\eta x)\}}{V_\alpha\{I_0(x),H_0^{(2)}(\eta x)\}}\right.\\ &\quad\left.+f(-ix)\frac{V_\alpha\{K_0(x),H_0^{(1)}(\eta x)\}}{V_\alpha\{I_0(x),H_0^{(1)}(\eta x)\}}\right],\end{aligned} \tag{22.16}$$

where s_0 is defined by the relation $k_{s_0} < x_0 < k_{s_0+1}$. This formula is valid for functions $f(z)$ obeying the condition

$$|f(z)| < \epsilon(x)e^{c|y|},\ z=x+iy,\ |z|\to\infty, \tag{22.17}$$

where $c < 2$ and $\epsilon(x) \to 0$ for $x \to \infty$. Formula (22.16) is further simplified for functions satisfying the additional condition $f(-ix) = -f(ix)$:

$$\lim_{x_0\to\infty}\left[\sum_{s=1}^{s_0}P_\alpha(k_s)f(k_s)-\int_0^{x_0}dx\,f(x)\right] = -\frac{4i\alpha}{\pi^2}\int_0^\infty dx\,\frac{f(ix)}{x^2 g_\alpha(\eta,x)}, \tag{22.18}$$

where

$$\begin{aligned}g_\alpha(\eta,x) &= I_0^2(x)\left[J_1^2(\eta x)+Y_1^2(\eta x)\right]+\alpha^2\eta^2 I_1^2(x)\left[J_0^2(\eta x)+Y_0^2(\eta x)\right]\\ &\quad -2\alpha\eta I_0(x)I_1(x)\left[J_0(\eta x)J_1(\eta x)+Y_0(\eta x)Y_1(\eta x)\right].\end{aligned} \tag{22.19}$$

Note that we have denoted $g_\alpha(\eta,x) = |V\{I_0(z),H_0^{(1)}(\eta z)\}|^2$ and this function is always non-negative.

Now, summation formulae for the series over $m=0$ TM and TE modes are obtained taking

$$\eta = \sqrt{\frac{1-\beta_{1\|}^2}{\beta_{0\|}^2-1}}. \tag{22.20}$$

In formula (22.16) we choose $\alpha = \varepsilon_0/\varepsilon_1$, $f(z) = zJ_0^2(z\rho_0/\rho_1)$ for the waves of the E-type, and $\alpha = 1$, $f(z) = zJ_1^2(z\rho_0/\rho_1)$ for the waves of the M-type. For both types of modes one has $f(-ix) = -f(ix)$ and we can use the version of the summation formula given by (22.18). In the intermediate step of the calculations, it is technically simpler instead of considering the dispersion

of the dielectric permittivity to assume that in formulae (22.6), (22.7) a cutoff function $\psi_\mu(\lambda_s^{(\mathrm{F})})$ is introduced with μ being the cutoff parameter and $\psi_0 = 1$ (for example, $\psi_\mu(x) = \exp(-\mu x)$), which will be removed after the summation. In this way, after the application of the summation formula we find the following results

$$I_{m=0}^{(\mathrm{E})} = q^2 v_\parallel \left[c^2 \int_0^\infty dx \frac{x}{\varepsilon_0} \psi_\mu(x) J_0^2(\rho_0 x) + \frac{4}{\pi^2 \rho_1^2} \int_0^\infty dx \, \frac{I_0^2(x\rho_0/\rho_1)}{\varepsilon_1 x g_{\varepsilon_0/\varepsilon_1}(\eta, x)} \right], \tag{22.21}$$

$$I_{m=0}^{(\mathrm{M})} = \frac{q^2 v_\perp^2 v_\parallel}{c^2} \left[\int_0^\infty dx \frac{x\psi_\mu(x)}{\beta_{0\parallel}^2 - 1} J_1^2(\rho_0 x) - \frac{4}{\pi^2 \rho_1^2} \int_0^\infty dx \, \frac{I_1^2(x\rho_0/\rho_1)}{(\beta_{0\parallel}^2 - 1) x g_1(\eta, x)} \right], \tag{22.22}$$

where the function $g_\alpha(\eta, x)$ is defined by formula (22.19). In the first terms in the square brackets replacing the integration variable by the frequency $\omega = v_\parallel x/\sqrt{\beta_{0\parallel}^2 - 1}$ and introducing the physical cutoff through the condition $\beta_{0\parallel} > 1$ instead of the cutoff function, we see that these terms coincide with the radiation intensities on the harmonic $m = 0$ for the waves of the E- and M-type in the homogeneous medium with dielectric permittivity ε_0. The second terms in the square brackets are induced by the inhomogeneity of the medium in the problem under consideration. Note that, unlike to the terms corresponding to a homogeneous medium, for $\rho_0 < \rho_1$ the terms induced by the inhomogeneity are finite also in the case when the dispersion is absent: for large values of x the integrands decay as $\exp[-2x(1 - \rho_0/\rho_1)]$ (for this reason we have removed the cutoff function from these terms). In particular, from the last observation it follows that under the condition $(1 - \rho_0/\rho_1) \gg v_\parallel/\omega_d$, where ω_d is the characteristic frequency for the dispersion of the dielectric permittivity, the influence of the dispersion on the inhomogeneity induced terms can be neglected. Note that in a homogeneous medium the corresponding radiation propagates under the Cherenkov angle $\theta_C = \arccos(1/\beta_{0\parallel})$ and has a continuous spectrum, whereas the radiation described by (22.21), (22.22) propagates inside the dielectric cylinder and has a discrete spectrum with frequencies $\omega_s^{(\mathrm{F})}$. As the function $g_\alpha(\eta, x)$ is always non-negative, from formulae (22.21),(22.22) we conclude that the presence of the cylinder amplifies the $m = 0$ part of the radiation for the waves of the E-type and suppresses the the radiation for the waves of the M-type. In the helical undulators one has $v_\perp \ll v_\parallel$ and the contribution of the TM waves dominates.

23 Summary

The Abel-Plana summation formula is a powerful tool for the evaluation of the difference between a sum and the corresponding integral. This formula has found numerous physical applications including the Casimir effect for various bulk and boundary geometries. However, the applications of the APF in its standard form are restricted to the problems where the normal modes are explicitly known. In the present paper we have considered a generalization of the APF, proposed in [27], which essentially enlarges the application range and allows to include problems where the eigenmodes are given implicitly as zeros of a given function. Well known examples of this kind are the boundary-value problems with spherical and cylindrical boundaries. The generalized version contains two meromorphic functions $f(z)$ and $g(z)$ and is formulated in the form of Theorem 1. The special choice $g(z) = -if(z)\cot \pi z$ gives the APF with additional residue terms coming from the poles of the function $f(z)$. We have shown that various generalizations of the APF previously discussed in literature are obtained from the GAPF by specifying the function $g(z)$. Further we consider applications of the GAPF to cylinder functions. These applications include summation formulae for series over the zeros of these functions and their combinations (Sections 3,4,5) and formulae for integrals involving cylinder functions (Sections 6,7,8). In the second part

of the paper, including Sections 9–22, we outline the applications of the summation formulae over the zeros of cylinder functions for the evaluation of the VEVs for local physical observables in the Casimir effect and the radiation intensity from a charge moving along a helical trajectory inside a dielectric cylinder.

In Section 3, in the GAPF choosing the function $g(z)$ in the form (3.2), we derive two types of summation formulae, (3.18), (3.33), for the series over the zeros $\lambda_{\nu,k}$ of the function $AJ_\nu(z) + BzJ'_\nu(z)$. This type of series arises in a number of problems of mathematical physics with spherically and cylindrically symmetric boundaries. As a special case they include Fourier-Bessel and Dini series (see [64]). Using formula (3.18), the difference between the sum over zeros $\lambda_{\nu,k}$ and the corresponding integral is presented in terms of an integral involving modified Bessel functions plus residue terms. For a large class of functions $f(z)$ the last integral converges exponentially fast and, in particular, is useful for numerical calculations. The APF is a special case of formula (3.18) with $\nu = 1/2$, $A = 1$, $B = 0$ and for an analytic function $f(z)$. Choosing in (3.18) $\nu = 1/2$, $A = 1$, $B = 2$ we obtain the APF in the form (2.19) useful for fermionic field calculations. Note that formula (3.18) may also be used for some functions having poles and branch points on the imaginary axis. The second type of summation formula, (3.33), is valid for functions satisfying condition (3.27) and presents the difference between the sum over zeros $\lambda_{\nu,k}$ and the corresponding integral in terms of residues over poles of the function $f(z)$ in the right half-plane (including purely imaginary ones). This formula may be used to summarize a large class of series of this type in finite terms. In particular, the examples we found in literature, when the corresponding sum may be presented in a closed form, are special cases of this formula. A number of new series summable by this formula and some classes of functions to which it can be applied is presented.

As a next application of the GAPF, in Section 4 we consider series over zeros $z = \gamma_{\nu,k}$ of the function $\bar{J}_\nu^{(a)}(z)\bar{Y}_\nu^{(b)}(\eta z) - \bar{Y}_\nu^{(a)}(z)\bar{J}_\nu^{(b)}(\eta z)$, where the barred notations are defined by relation (4.2). The corresponding results are formulated in the form of Corollary 2 and Corollary 3. Using formula (4.14), the difference between the sum over $\gamma_{\nu,k}$ and the corresponding integral can be expressed as an integral containing modified Bessel functions plus residue terms. For a large class of functions $h(z)$ this integral converges exponentially fast. The formula of the second type, (4.18), allows to summarize a class of series over the zeros $\gamma_{\nu,k}$ in closed form. To evaluate the corresponding integral, the formula can be used derived in section 8. This yields to another summation formula, (4.21), containing residue terms only. The examples we have found in literature when the corresponding sum was evaluated in closed form are special cases of the formulae considered here. We present new examples and some classes of functions satisfying the corresponding conditions. In Section 5, taking in the GAPF the functions in the form (5.3), we derive summation formula (5.12) for series over the zeros $z = \omega_k$ of the function $AK_{iz}(\eta) + B\eta\partial_\eta K_{iz}(\eta)$. These zeros are eigenfrequencies for scalar and electromagnetic fields in the geometry of a uniformly accelerated plane boundary and the corresponding VEVs contain summation over them. For the geometry of two plane boundaries the eigenfrequencies in the region between the planes are zeros $z = \Omega_k$ of the function $\bar{K}_{iz}^{(a)}(u)\bar{I}_{iz}^{(b)}(v) - \bar{I}_{iz}^{(a)}(u)\bar{K}_{iz}^{(b)}(v)$. Summation formula (5.18) for series over these zeros is obtained from (2.11) with the functions $f(z)$ and $g(z)$ given by (5.15).

In Section 6 we consider relations for the integrals of type

$$\int_0^\infty F(x)\bar{J}_\nu(x)dx, \ \int_0^\infty F(x)[J_\nu(x)\cos\delta + Y_\nu(x)\sin\delta]dx,$$

which are obtained from the GAPF. The corresponding formulae have the form (6.4), (6.7) and (6.18). In particular, formula (6.4) is useful to express the integrals containing Bessel functions with oscillating integrand through the integrals of modified Bessel functions with exponentially

fast convergence. Formulae (6.7) and (6.18) allow to evaluate a large class of integrals involving the Bessel function in closed form. In particular, the results obtained in [68] are special cases of these formulae. Illustrating examples of applications of the formulae for integrals are given in Section 7 (see (7.3)-(7.7) and (7.13)-(7.17)). Looking at the standard books (see, e.g., [13], [64]-[72]) one will find many particular cases which follow from these formulae. A number of new integrals can be evaluated as well. We also consider two examples of functions having purely imaginary poles, (7.19) and (7.23), with corresponding formulae (7.20) and (7.24) (for two special cases of these formulae see [64]). By choosing the functions $f(z)$ and $g(z)$ in accordance with (8.1), formulae (8.7) and (8.16) for the integrals of the type

$$\int_0^\infty \frac{J_\nu(x)Y_\mu(\lambda x) - J_\mu(\lambda x)Y_\nu(x)}{J_\nu^2(x) + Y_\nu^2(x)} F(x)dx$$

can be derived from the GAPF. The corresponding results are formulated in the form of Theorem 5 and Theorem 6 in Section 8. Several examples for integrals of this type we have been able to find in literature are particular cases of formula (8.7). New examples when the integral is evaluated in finite terms are presented. Some classes of functions are distinguished to which the corresponding formulae may be applied.

In the second part of the paper, started from Section 9, physical applications of the summation formulae for series over zeros of cylinder functions obtained from the GAPF are reviewed. The general procedure for the evaluation of the VEVs of the field square and the EMT on manifolds with boundaries based on the point-splitting technique is described. This VEVs are obtained from the corresponding positive frequency Wightman function in the coincidence limit by using formulae (9.9), (9.10). In these formulae instead of the Wightman function any other two-point function can be chosen. Our choice of the Wightman function is related to that it also determines the response of the Unruh-DeWitt type particle detectors. Mode-sum (9.8) contains a summation over the eigenmodes of the problem under consideration. For curved boundaries these eigenmodes are given implicitly as zeros of the eigenfunctions for the corresponding boundary value problem or their combinations. The second difficulty for the direct evaluation of the mode-sum is related to that it diverges in the coincidence limit and exhibits very slow convergence characteristics when the arguments of the Wightman function are close to each other. In addition, the separate terms in the mode-sum are strongly oscillatory for higher modes. The application of summation formulae obtained from the GAPF enables to present the mode-sum in terms of integrals and, hence, in the corresponding procedure the explicit form of the eigenmodes is not necessary. These formulae explicitly extract from the VEVs of local physical observables the parts corresponding to the bulk without boundaries and the boundary-induced parts are presented in terms of integrals which are exponentially convergent for the points away from the boundaries. As a result, in the coincidence limit the renormalization is necessary for the boundary-free parts only and this procedure is the same as that in quantum field theory without boundaries. Note that, by using summation formulae obtained from the GAPF, the mode-sums for more general class of objects such as the heat and cylinder kernels can be evaluated as well.

First, in Section 10 we consider examples where the eigenmodes are explicitly known and the application of the APF in standard form enables to obtain the renormalized VEVs. These examples include the problems for the evaluation of the VEVs in topologically non-trivial space $R^D \times S^1$ and for the geometry of two parallel plates with Dirichlet and Neumann boundary conditions on them. On these simple examples we have seen two important advantages of the application of the APF. First, this formula enables to extract explicitly from the VEVs the Minkowskian part and second, to present the parts induced by the non-trivial topology/boundaries in terms of rapidly convergent integrals. Already in the case of two parallel plates with Robin boundary conditions the eigenmodes are not explicitly known and the procedure for the evaluation of the

VEVs based on the APF needs a generalization. The corresponding problem is discussed in Section 11. In this problem the eigenmodes for the projection of the wave vector perpendicular to the plates are solutions of transcendental equation (11.3). For the summation of the mode-sums over these eigenmodes we have derived formula (11.7) by making use of the GAPF. The application of this summation formula enables to extract from the Wightman function the part corresponding to the geometry of a single plate (see formula (11.9)). Similar decomposed formulae are obtained for the VEVs of the field square and the EMT.

In Section 12 we consider the scalar Casimir densities inside a spherical boundary and in the region between two concentric spherical boundaries on background of global monopole spacetime described by line element (12.1). It is assumed that on the bounding surfaces the scalar field obeys Robin boundary conditions. For the region inside a single sphere the eigenmodes are zeros of the function $\bar{J}_{\nu_l}(x)$, where the order of the Bessel function is defined by formula (12.4) and the coefficients in the barred notation are related to the coefficients in the boundary condition by formula (12.6). For the evaluation of the corresponding mode-sum we have used summation formula (3.18). The term with the integral on the left of this formula corresponds to the Wightman function for the global monopole bulk without boundaries and the term with the integral on the right is induced by the spherical boundary. In the region between two spheres the eigenmodes are zeroes of the function $C_{\nu}^{ab}(\eta, z)$ defined by formula (4.1) and for the summation over these eigenmodes formula (4.14) is used. We have seen that the term with the integral on the left of this formula corresponds to the Wightman function in the region outside a single sphere and the term with the integral on the right is induced by the presence of the outer sphere.

In section (13) we have investigated the vacuum densities for a fermionic field induced by spherical boundaries in a global monopole background. It is assumed that the field obeys bag boundary condition. Both regions inside a single sphere and between two spheres are considered. For the first case the eigenmodes are zeros of the function $\tilde{J}_{\nu_\sigma}(x)$ with the tilted notation defined by (13.7). By special choice of the function $g(z)$ in the GAPF we have derived formula (13.17) for the summation over these zeros. The application of this formula to the mode-sum for the VEVs enables to extract the parts corresponding to the boundary-free bulk without specifying the form of the cutoff function. In the region between two spheres the eigenmodes are zeros of the function $C_{\nu_\sigma}^{\mathrm{f}}(\eta, x)$ defined by formula (13.29). By special choice of the functions $f(z)$ and $g(z)$ in the GAPF given by (13.35), in subsection 13.2 we have derived summation formula (13.42) for the series over these zeros. This formula presents the VEV of the EMT in the form of the sum of a single sphere and second sphere induced parts. Scalar and fermionic Casimir densities for spherical boundaries in the Minkowski bulk are obtained from the corresponding results for the global monopole geometry as special cases with $\alpha = 1$. In Section 14, by making use of the GAPF we have investigated the local properties of the electromagnetic vacuum inside a perfectly conducting spherical shell and in the region between two spherical shells on background of Minkowski spacetime.

Vacuum polarization by cylindrical boundaries is considered in Sections 15, 16, 17. We start with the case of a scalar field on background of the cosmic string spacetime with line element (15.1) and with coaxial cylindrical boundary. On the boundary the field obeys Robin boundary condition. In the region inside the cylindrical shell the eigenmodes are zeros of the function $\bar{J}_{q|n|}(z)$, where the parameter q is determined by the planar angle deficit and the coefficients in the barred notation are defined by relation (15.5). The application of the GAPF to the mode-sum of the Wightman function extracts the part corresponding to the cosmic string geometry without boundaries and the boundary induced part is presented in the form which can be directly used for the evaluation of the VEVs for the field square and the EMT. In Section 16 similar consideration is done for the electromagnetic field with perfectly conducting boundary conditions. The corresponding eigenmodes are the zeros of the function $J_{q|n|}(z)$ for the TM

waves and the zeros of the function $J'_{q|n|}(z)$ for the TE waves. The formulae for the VEVs induced by a cylindrical boundary in the Minkowski bulk are obtained from the corresponding results for the cosmic string geometry in the special case $q = 1$ with zero planar angle deficit. Vacuum densities for both scalar and electromagnetic fields in the region between two coaxial cylindrical shells in background of Minkowski spacetime are investigated in Section 17.

The use of general coordinate transformations in quantum field theory in flat spacetime leads to an infinite number of unitary inequivalent representations of the commutation relations with different vacuum states. In particular, the vacuum state for a uniformly accelerated observer, the Fulling-Rindler vacuum, turns out to be inequivalent to that for an inertial observer, the Minkowski vacuum. In Section 18, the polarization of the Fulling-Rindler vacuum for a scalar field by a uniformly accelerated plate with Robin boundary conditions is investigated by using the GAPF. The corresponding eigenfrequencies are the zeros of the function $\bar{K}_{i\omega}(x)$, where the coefficients in the barred notation (5.1) are related to the Robin coefficients by formula (18.4). On the base of the summation formula derived in Section 5 for the series over these zeros, we have extracted from the Wightman function the part corresponding to the Rindler wedge without boundaries. The part induced by the plate is given by formula (18.10) and is directly used for the evaluation of the VEVs for the field square and the EMT. In the case of $D = 3$ electromagnetic field with perfect conductor boundary condition on the plate, the corresponding eigenfunctions for the vector potential are resolved into the TE and TM scalar modes with Dirichlet and Neumann boundary conditions respectively. The corresponding VEV of the EMT is given by formula (18.26). The Casimir densities in the region between two infinite parallel plates moving by uniform proper accelerations are discussed in Section 19 for both scalar and electromagnetic fields. In the case of scalar field the eigenfrequencies are the zeros $\omega = \Omega_n$ of the function $Z_{i\omega}(u, v)$ defined by formula (5.14) (see (19.3)). After the application of summation formula (5.18), the Wightman function is presented in the form (19.7) or equivalently (19.12). The first term on the right of these formulae is the Wightman function for the case of a single plate and the second term is induced by the presence of the second plate. Similar decomposition is obtained for the VEVs of the field square and the EMT, formulae (19.13) and (19.19). We have investigated the parts in the VEVs induced by the presence of the second brane in various asymptotic limits of the parameters. The vacuum interaction forces between the plates are considered as well. The VEV of the EMT for the electromagnetic field in the region between the plates is given by formula (19.30) or equivalently by (19.31), and the corresponding interaction forces are given by formulae (19.32), (19.33).

Recent proposals of large extra dimensions use the concept of brane as a sub-manifold embedded in a higher dimensional spacetime, on which the Standard Model particles are confined. Braneworlds naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions. In Section 20 we consider the geometry of two parallel flat branes in the AdS bulk. In the region between the branes the radial Kaluza-Klein masses $m = m_n$ are zeros of the function $\bar{J}_\nu^{(a)}(mz_a)\bar{Y}_\nu^{(b)}(mz_b) - \bar{Y}_\nu^{(a)}(mz_a)\bar{J}_\nu^{(b)}(mz_b)$, where the coefficients in barred notation (4.2) are related to the Robin coefficients by formulae (20.8). The application of summation formula (4.14) allows to extract from the VEVs the part due to a single brane. Both single brane and second brane induced parts and vacuum interaction forces between the branes are investigated. An application to the Randall-Sundrum braneworld with arbitrary mass terms on the branes is discussed. Similar issues for two spherical branes in Rindler-like spacetimes are considered in Section 21. The corresponding eigenfrequencies $\omega = \Omega_n$ are zeros of the function $\bar{K}_{i\omega}^{(a)}(\lambda_l a)\bar{I}_{i\omega}^{(b)}(\lambda_l b) - \bar{I}_{i\omega}^{(a)}(\lambda_l a)\bar{K}_{i\omega}^{(b)}(\lambda_l b)$, where a and b are the branes radii and λ_l is defined by (21.6). In Section 22 we apply the GAPF for the investigation of the radiation intensity from a charge moving along a helical trajectory inside a dielectric cylinder. Summation formula

(22.16) is derived for the series over the zeros of function (22.9). This formula is used for the summation of the series over the TE and TM eigenmodes of the dielectric cylinder appearing in the expressions for the corresponding radiation intensities on the lowest azimuthal harmonic.

Of course, the applications of the summation formulae obtained from GAPF are not restricted by the Casimir effect only. Similar types of series will arise in considerations of various physical phenomenon near the boundaries with spherical and cylindrical symmetries, for example in calculations of the electron self-energy and the electron anomalous magnetic moment (for similar problems in the plane boundary case see, e.g., [146] and references therein). The dependence of these quantities on boundaries originates from the modification of the photon propagator due to the boundary conditions imposed by the walls of the cavity.

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